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Rapid stabilization of an unstable heat equation with disturbance at the flux boundary condition

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ABSTRACT

In this paper we prove the rapid stabilization of an unstable heat equation subjected to an unknown disturbance, which is assumed to be acting at the flux boundary condition. To that end, we design a multivalued feedback law by employing the backstepping method, Lyapunov techniques and the sign multivalued operator, which is used to handle the effects of the unknown disturbance. The well-posedness of the closed-loop system, which is a differential inclusion, is shown with the maximal monotone operator theory.

1. Introduction

Partial differential equations have been widely and successfully used to derive mathematical models for diverse phenomena, such as the temperature on solids or fluids, the propagation of waves in a medium, the lateral deflection of strings or beams, to name a few remarkable examples. Once a mathematical model is established one relevant task in control theory is to design feedback laws to stabilize the state of the system to their equilibria or to another state of interest. In the literature we can frequently find that mathematical models are analyzed under ideal assumptions in which disturbances are neglected for the sake of simplicity, but disturbances are always present and indeed may correspond to an additional source of instability. Accordingly, it is relevant to include disturbances in the study of the stabilization problem.

In this paper we address the problem of boundary stabilization of an unstable heat equation subjected to an unknown disturbance acting at the flux boundary condition. Let $L \in (0,\infty)$ and $a \in C^1([0,L])$. Let us consider the system described by

$$\begin{cases} z_t - z_{xx} = az, & (t,x) \in (0,\infty)\times(0,L), \\ z_x(t,0) = 0, & t \in (0,\infty), \\ z_x(t,L) = u(t) + d(t), & t \in (0,\infty), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{1.1}$$

For a (regular enough) state $z = z(t,x)$ of (1.1) we define its energy by

$$E(t) = \frac{1}{2}\int_0^L |z|^2\,dx,\ \ t \in [0,\infty). \tag{1.2}$$

The purpose of this paper is to exponentially stabilize the (regular enough) state $z = z(t,x)$ of (1.1) to the rest by means of a feedback law $u(t)$ that suppresses the effects of an unknown disturbance $d(t)$. Being precise, we aim to prove the rapid stabilization of (1.1), which in this paper is understood as follows:

$$\begin{cases} \text{For any desired decay rate } \omega \in (0,\infty) \\ \text{there exists a feedback law } u(t) \text{ and there exists a constant} \\ \quad C \in [1,\infty) \\ \text{such that } E(t) \le Ce^{-2\omega t}E(0) \text{ for all } t \in [0,\infty). \end{cases} \tag{1.3}$$

Regarding the undisturbed case, which is when the disturbance is zero, the sources of instability of (1.1) come from its boundary conditions and $a^+(x) = \max\{a(x),0\}$ (the non-negative part of a). In this case the problem under consideration, which is the obtention of (1.3), has been successfully solved in [1] with a feedback law designed by means of the backstepping method and Lyapunov techniques. Such a feedback law is given by [1, (3.3)] and reads as

$$u(t) = -k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,ds, \tag{1.4}$$

where the gain kernel $k = k(x,s)$ is a C^2 function on the triangle $\Omega = \{(x,s) \in \mathbb{R}^2 \ /\ 0 \le s \le x \le L\}$ uniquely solving [1, (3.1)], that is to say,

$$\begin{cases} k_{xx}(x,s) - k_{ss}(x,s) = (a(s)+\omega)k(x,s), & (x,s) \in \Omega, \\ k_s(x,0) = 0, & x \in [0,L], \\ k(x,x) = \frac{1}{2}\int_0^x (a(s)+\omega)\,ds, & x \in [0,L]. \end{cases} \tag{1.5}$$

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Table 1
Stabilization of partial differential equations subjected to unknown disturbances; meeting similar assumptions as **(A1)** and **(A2)** given below.

Equation	Distributed disturbance	Boundary disturbance
Heat	[2]	[3–5]
Wave	[6]	[7–9]
Beam	[10]	[11–13]
Schrödinger	–	[14,15]

Regarding the disturbed case, which is the case we address in this paper, it is uncertain whether we can employ the same feedback law (1.4) to solve the problem under consideration, which is the obtention of (1.3), since in the construction of the gain kernel $k = k(x,s)$ by uniquely solving (1.5) no assumption on the unknown disturbance is used, and hence, the same feedback law (1.4) might not be able to handle its effects. Indeed:

Claim 1.1. *If the disturbance is not zero, then the feedback law (1.4) does not longer work.*

Proof. For the sake of the argument, in (1.1) let us consider $L = \pi/2$, the constant parameter $a(x) = 1$, the feedback law (1.4), the constant disturbance $d(t) = -\sin(L) + k(L,L)\cos(L) + \int_0^L k_x(L,s)\cos(s)\,ds$ and the initial condition $z_0(x) = \cos(x)$. Then, the corresponding unique solution to (1.1) is $z(t,x) = \cos(x)$. Since in this case we get that $E(t) = (1/2)\sin(L) = 1/2$, it follows that (1.3) cannot be satisfied. ■

Accordingly, in (1.1) we may regard the unknown disturbance as another source of instability and a new feedback law is required to solve the problem under consideration. In this paper we propose a solution to this open problem.

The stabilization problem for partial differential equations subjected to unknown disturbances, acting either in the domain or at the boundary, has been object of recent interest. In Table 1 we present, without being exhaustive, some of the concerned literature.

In the works presented in Table 1, the effects of the unknown disturbances were handled with the aid of the sign multivalued operator. In our case, we consider the sign multivalued operator $\text{sign} : \mathbb{R} \to 2^{\mathbb{R}}$ ($2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$) given by

$$\text{sign}(f) = \begin{cases} \dfrac{f}{|f|} & \text{if} \quad f \neq 0, \\ [-1,1] & \text{if} \quad f = 0. \end{cases} \tag{1.6}$$

Regarding the disturbance, although it is assumed unknown, we ask it to satisfy the following two assumptions, which are the standard ones that can be found in the literature.

(A1) There exists $D \in (0,\infty)$ such that $|d(t)| \leq D$ for every $t \in [0,\infty)$.

(A2) $d \in W^{2,1}(0,\infty)$ and $d(0) = 0$.

On the one hand, assumption **(A1)** is required for the construction of a feedback law able to reject the effects of the disturbance. In assumption **(A1)** the constant $D \in (0,\infty)$ can be chosen as we see fit: the greater its value is, the more disturbances the feedback law will be able to reject. On the other hand, assumption **(A2)** is required to prove the well-posedness of the corresponding closed-loop system.

In order to present our main result, we need to introduce some elements. First, let us consider the backstepping transformation $K : H^2(0,L) \to H^2(0,L)$ given by

$$(Kz_0)(x) = z_0(x) + \int_0^x k(x,s)z_0(s)\,ds = y_0(x),$$

where $k = k(x,s)$ is the gain kernel obtained from (1.5). In virtue of [1, Lemma 3.3] it follows that K has a linear and bounded inverse $K^{-1} : H^2(0,L) \to H^2(0,L)$. Also, let us consider the set

$$\mathcal{I} = \{y_0 \in H^2(0,L) \ / \ y_0'(0) = 0 \text{ and } y_0'(L) + D\text{sign}(y_0(L)) \ni 0\}.$$

Let us note that $\mathcal{I} \subset H^2(0,L)$ and that $\mathcal{I}$ is not an empty set since $H_0^2(0,L) \subset \mathcal{I}$. Indeed, for any $y_0 \in H_0^2(0,L)$ we get the true statement $[-D,D] \ni 0$.

Our main result is the following one.

Theorem 1.1. *Let $a \in C^1([0,L])$. Let $\omega \in (0,\infty)$ be the desired decay rate and let $k = k(x,s)$ be the gain kernel obtained from (1.5). Let us assume **(A1)** and **(A2)**. For a regular enough function $f = f(t,x)$ let us introduce the multivalued feedback law*

$$u(t,f) = -k(L,L)f(t,L) - \int_0^L k_x(L,s)f(t,s)\,ds - D\text{sign}\left(f(t,L) + \int_0^L k(L,s)f(t,s)\,ds\right). \tag{1.7}$$

Let $z_0 \in K^{-1}(\mathcal{I}) = \{K^{-1}y_0 \ / \ y_0 \in \mathcal{I}\}$ be an initial condition. Then, there exists a unique $z = z(t,x)$ in $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ such that

$$\begin{cases} z_t - z_{xx} = az \text{ for almost every } (t,x) \in (0,\infty) \times (0,L), \\ z_x(t,0) = 0 \text{ for every } t \in [0,\infty), \\ z_x(t,L) \ni u(t,z) + d(t) \text{ for every } t \in [0,\infty), \\ z(0,x) = z_0(x) \text{ for every } x \in (0,L). \end{cases} \tag{1.8}$$

Moreover, there exists a constant $C \in [1,\infty)$ such that

$$\|z(t,\cdot)\|_{L^2(0,L)} \leq Ce^{-\omega t}\|z_0\|_{L^2(0,L)} \text{ for all } t \in [0,\infty). \tag{1.9}$$

Remark 1.1. Since (1.9) is valid for any desired decay rate $\omega \in (0,\infty)$, we have obtained the rapid stabilization (1.3) of the unstable heat Eq. (1.1). Let us note that except by [15], the known results in the stabilization of partial differential equations subjected to disturbances at the boundary are not rapid, in the sense described in (1.3). This is an improvement of our previous work [5], in which a heat equation with variable coefficients and boundary disturbance is addressed.

Remark 1.2. The closed-loop system (1.8) is a differential inclusion. Regarding differential inclusions and its well-posedness, the interested reader may consult the papers [16–19] and the books [20–22].

Remark 1.3. The assumption $z_0 \in K^{-1}(\mathcal{I})$ for the initial condition is needed in order to have regular enough solutions to the closed-loop system (1.8); assumption that is similar in nature to [1, (3.6)]. Such an assumption is a technicality coming from the application of the backstepping method.

In Theorem 1.1 the multivalued feedback law (1.7) is composed by two parts. The first part, which actually is (1.4), is needed to achieve the desired decay rate; while the second part uses the sign multi valued operator (1.6) to handle the effects of the unknown disturbance. Overall, the multivalued feedback law (1.7) is designed by employing the backstepping method and Lyapunov techniques. The backstepping method (see [23,24] for instance) has shown to be useful for solving rapid stabilization problems, as can be consulted in [1,15,25–31] for instance. The proof of the feedback design part described here is done in Section 2. In Theorem 1.1 the corresponding closed-loop system is the differential inclusion (1.8) and its well-posedness is shown with the maximal monotone operator theory, which may be consulted in [20–22] for instance. The proof of the well-posedness part described here is done in Section 3.

In Section 4 we provide final comments and propose an open problem.

Remark 1.4. As described above, the idea behind the feedback design is to split the control into two parts, using each part for different purposes. This idea is one of the main contributions of this paper, besides the rapid stabilization result (Remark 1.1).

Remark 1.5. The approach we present in this paper does not require the introduction of a sliding surface nor of a sliding variable. In fact, the control we design, given by (1.7), apparently is not a sliding mode control, but is similar in the sense that we also make use of the sign multivalued operator (1.6). The sliding mode control procedure may be consulted in [4,8,14,32–34] for instance.

2. Feedback design

In this section we employ the backstepping method, Lyapunov techniques and the sign multivalued operator (1.6) to obtain the multivalued feedback law (1.7). The main idea behind the feedback design proposed here is to split $u(t)$ as $u(t) = u_1(t) + u_2(t)$, where $u_1(t)$ is used to achieve the desired decay rate and $u_2(t)$ is used to reject the effects of the unknown disturbance.

Let us assume the hypotheses of Theorem 1.1. By [1, Lemma 3.2] we have that (1.5) has a unique solution $k = k(x,s)$, which is a C^2 function on the triangle $\Omega = \{(x,s) \in \mathbb{R}^2 \ / \ 0 \le s \le x \le L\}$. Then, let us consider the backstepping transformation

$$y(t,x) = z(t,x) + \int_0^x k(x,s)z(t,s)\,ds \tag{2.1}$$

and let us take

$$u_1(t) = -k(L,L)z(t,L) - \int_0^L k_x(L,s)z(t,s)\,ds. \tag{2.2}$$

Accordingly, thanks to the proof of [1, Theorem 3.1], which follows the same arguments as in the proof of [1, Theorem 2.1], we have that (1.1) is transformed into

$$\begin{cases} y_t - y_{xx} + \omega y = 0, & (t,x) \in (0,\infty)\times(0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) = u_2(t) + d(t), & t \in (0,\infty), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{2.3}$$

where $y_0(x) = z_0(x) + \int_0^x k(x,s)z_0(s)\,ds$. Let us assume that $y = y(t,x)$ is a regular enough solution of (2.3). We proceed to perform formal computations, which later are justified in view of the analysis done in the next section.

From (2.3) it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |y|^2 dx\right) + \int_0^L |y_x|^2 dx - y_x y\Big|_{x=0}^{x=L} + \omega \int_0^L |y|^2 dx = 0,$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |y|^2 dx\right) + \omega \int_0^L |y|^2 dx \le (u_2(t) + d(t))y(t,L). \tag{2.4}$$

Then, with the choice of

$$u_2(t) = -D\mathrm{sign}(y(t,L)) \tag{2.5}$$

the right-hand side of (2.4) becomes non-positive. Indeed, thanks to the property $\theta p = |p|$ for every $\theta \in \mathrm{sign}(p)$ and Assumption **(A1)** we have

$$\begin{aligned}(u_2(t) + d(t))y(t,L) &\le -D\theta y(t,L) + |d(t)||y(t,L)| \\ &= -D|y(t,L)| + D|y(t,L)| = 0 \text{ for every} \\ &\quad \theta \in \mathrm{sign}(y(t,L)).\end{aligned} \tag{2.6}$$

Therefore, in virtue of (2.4) and (2.6) we conclude that the solutions of the corresponding closed-loop system, obtained by plugging (2.5) into (2.3), satisfy

$$\|y(t,\cdot)\|^2_{L^2(0,L)} \le e^{-2\omega t}\|y_0\|^2_{L^2(0,L)} \text{ for all } t \in [0,\infty). \tag{2.7}$$

Remark 2.1. We get the multivalued feedback law (1.7) by considering that $u(t) = u_1(t) + u_2(t)$, that $u_1(t)$ is the one given by (2.2) and that $u_2(t)$ is the one given by (2.5) with (2.1) evaluated at $x = L$.

In order to conclude (1.9) we use (2.7) and that both the backstepping transformation and its inverse are linear and bounded operators. Indeed, by [1, Lemma 3.3] we have that the backstepping transformation $K : L^2(0,L) \to L^2(0,L)$ given by

$$(Kf)(x) = f(x) + \int_0^x k(x,s)f(s)\,ds,$$

has a linear and bounded inverse $K^{-1} : L^2(0,L) \to L^2(0,L)$. Let us set $C_1 = \|K^{-1}\|_{\mathcal{L}(L^2(0,L))}$ and $C_2 = \|K\|_{\mathcal{L}(L^2(0,L))}$, which satisfy $C_1C_2 \ge 1$. Then, from (2.1) and (2.7) we get

$$\begin{aligned}\|z(t,\cdot)\|_{L^2(0,L)} &\le C_1\|y(t,\cdot)\|_{L^2(0,L)} \\ &\le C_1 e^{-\omega t}\|y_0\|_{L^2(0,L)} \\ &\le C_1C_2 e^{-\omega t}\|z_0\|_{L^2(0,L)} \text{ for all } t \in [0,\infty).\end{aligned}$$

We have shown the feedback design part of Theorem 1.1 and (1.9) with $C = C_1C_2$.

Remark 2.2. As in [24, Section 2.6], we mention that the inverse backstepping transformation is not used in the feedback design, but its existence and properties are indeed used for the proof of the closed-loop stability of (1.8).

3. Well-posedness

In this section we apply the maximal monotone operator theory to prove the well-posedness of the closed-loop system (1.8), which is a differential inclusion.

Let us recall that the backstepping transformation (2.1) maps (1.8) with (1.7) into (2.3) with $u_2(t)$ given by (2.5). The details are given in the proof of [1, Theorem 3.1], which follows the same arguments as in the proof of [1, Theorem 2.1]. Accordingly, we get

$$\begin{cases} y_t - y_{xx} + \omega y = 0, & (t,x) \in (0,\infty)\times(0,L), \\ y_x(t,0) = 0, & t \in (0,\infty), \\ y_x(t,L) + D\mathrm{sign}(y(t,L)) \ni d(t), & t \in (0,\infty), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{3.1}$$

where $y_0(x) = z_0(x) + \int_0^x k(x,s)z_0(s)\,ds$. With the introduction of $w(t,x) = y(t,x) - d(t)\phi(x)$, where $\phi(x) = (1/2L)x^2 - (L/2)$, it follows from (3.1) that $w = w(t,x)$ satisfies

$$\begin{cases} w_t - w_{xx} + \omega w = f, & (t,x) \in (0,\infty)\times(0,L), \\ w_x(t,0) = 0, & t \in (0,\infty), \\ w_x(t,L) + D\mathrm{sign}(w(t,L)) \ni 0, & t \in (0,\infty), \\ w(0,x) = w_0(x), & x \in (0,L), \end{cases} \tag{3.2}$$

where $f(t,x) = -d'(t)\phi(x) + d(t)\phi''(x) - \omega d(t)\phi(x)$ and $w_0(x) = y_0(x)$, the latter being in virtue of Assumption **(A2)**. Let us note that we have used: $\phi'(0) = 0$, $\phi'(L) = 1$ and $\phi(L) = 0$.

In order to perform the analysis, let us introduce the operator

$$\begin{cases} \mathcal{A} : D(\mathcal{A}) \subset L^2(0,L) \to L^2(0,L), \\ \mathcal{A}p = -p'' + \omega p, \\ D(\mathcal{A}) = \{p \in L^2(0,L) \ / \ \mathcal{A}p \in L^2(0,L),\ p'(0) = 0,\ p'(L) \\ \quad + D\mathrm{sign}(p(L)) \ni 0\}. \end{cases} \tag{3.3}$$

Due to the sign multivalued operator (1.6) it follows that $D(\mathcal{A})$ is not a linear subspace, and hence, $\mathcal{A}$ is not a linear operator. Thanks to (3.3) we can write (3.2) in operator form as follows:

$$\begin{cases} \dfrac{dw}{dt} + \mathcal{A}w = f, & t \in [0,\infty), \\ w(0) = w_0. \end{cases} \tag{3.4}$$

We proceed to prove that (3.4) is well-posed by applying the maximal monotone operator theory, which may be consulted in [20–22] for instance. In that direction we have the following two results: the first one (Proposition 3.1) states that the operator $\mathcal{A}$ is monotone, while the second one (Proposition 3.2) states that the operator $I+\mathcal{A}$ is surjective.

Proposition 3.1. *The operator defined by (3.3) is monotone.*

Proof. Let $(u,v)\in D(\mathcal{A})\times D(\mathcal{A})$. Then, we have

$$\begin{aligned}&(\mathcal{A}u-\mathcal{A}v,u-v)_{L^2(0,L)}\\&=\int_0^L(\mathcal{A}u-\mathcal{A}v)(u-v)\,dx\\&=\int_0^L\left[-(u-v)''+\omega(u-v)\right](u-v)\,dx\\&=\int_0^L(u'-v')^2\,dx+\omega\int_0^L(u-v)^2\,dx-\left(u'(L)-v'(L)\right)(u(L)-v(L)).\end{aligned}\tag{3.5}$$

Let us handle the last term of the right-hand side of (3.5). Since $(u,v)\in D(\mathcal{A})\times D(\mathcal{A})$, there exist $\widetilde{u}\in\text{sign}(u(L))$ and $\widetilde{v}\in\text{sign}(v(L))$ such that $u'(L)+D\widetilde{u}=0$ and $v'(L)+D\widetilde{v}=0$, thus obtaining

$$\left(u'(L)-v'(L)\right)(u(L)-v(L))=-D(\widetilde{u}-\widetilde{v})(u(L)-v(L)).\tag{3.6}$$

Finally, due to the monotonicity of the sign multivalued operator (1.6), we can combine (3.5) and (3.6) to conclude that $(\mathcal{A}u-\mathcal{A}v,u-v)_{L^2(0,L)}\geq 0$. Accordingly, the operator $\mathcal{A}$ is monotone. ■

Proposition 3.2. *The operator defined by (3.3) satisfies $R(I+\mathcal{A})=L^2(0,L)$.*

Proof. Given a $f\in L^2(0,L)$ we need to prove the existence of a $p\in D(\mathcal{A})$ such that $p+\mathcal{A}p=f$ for almost every $x\in(0,L)$, or equivalently, $p-p''+\omega p=f$ for almost every $x\in(0,L)$. To this end, we proceed as in [35, Section 2] and analyze an optimization problem (see also the proof of [5, Proposition 3.1]). Let us introduce the functional $J:H^1(0,L)\to\mathbb{R}$ by

$$J(p)=\frac{1}{2}\int_0^L\left[p^2+(p')^2+\omega p^2-2fp\right]dx+\varphi_\lambda(p(L)),\tag{3.7}$$

where $\varphi_\lambda:\mathbb{R}\to\mathbb{R}$ is the Moreau regularization of the convex and continuous function $\varphi:\mathbb{R}\to\mathbb{R}$ given by $\varphi(x)=D|x|$. Thanks to [21, Chapter IV, Proposition 1.8] (see also [22, Theorem 2.9]) we have

$$\varphi_\lambda(x)=\frac{\lambda}{2}|\alpha_\lambda(x)|^2+\varphi(J_\lambda(x)),\ \ (\lambda,x)\in(0,\infty)\times\mathbb{R},$$

where $\alpha_\lambda=\lambda^{-1}(I-J_\lambda):\mathbb{R}\to\mathbb{R}$ is the Yosida approximation of the maximal monotone operator $\alpha:\mathbb{R}\to 2^{\mathbb{R}}$ given by $\alpha(x)=(\partial\varphi)(x)=D\text{sign}(x)$ and $J_\lambda=(I+\lambda\alpha)^{-1}:\mathbb{R}\to\mathbb{R}$ is the resolvent of α. Furthermore, also from [21, Chapter IV, Proposition 1.8] (see also [22, Theorem 2.9]) we have that φ_λ is a convex and differentiable function satisfying that $\varphi_\lambda'(x)=\alpha_\lambda(x)$.

The functional J is convex and continuous, the latter being a consequence of the continuous injection of $H^1(0,L)$ into $C([0,L])$. Moreover, the functional J is coercive. Indeed, considering the non-negativity of φ_λ in (3.7) we get

$$J(p)\geq\frac{1}{2}\|p\|^2_{H^1(0,L)}-\|f\|_{L^2(0,L)}\|p\|_{H^1(0,L)},$$

from which we deduce the claim since $J(p)\to\infty$ as $\|p\|_{H^1(0,L)}\to\infty$. Therefore, [36, Theorem 2.19] yields the existence of a minimizer $m\in H^1(0,L)$ for the functional J. Then, in virtue of [36, Proposition 3.20] and [36, Theorem 3.24], we have that the Gâteaux derivative of the functional J at such minimizer must vanish for each direction $d\in H^1(0,L)$. Accordingly, as $\varphi_\lambda'(x)=\alpha_\lambda(x)$ we obtain

$$J'(m;d)=\int_0^L\left(md+m'd'+\omega md-fd\right)dx+\alpha_\lambda(m(L))d(L)=0\quad\forall d\in H^1(0,L).\tag{3.8}$$

In (3.8) we consider the facts $C_0^\infty(0,L)\subset H^1(0,L)$ and $m+\omega m-f\in L^2(0,L)$ to conclude that $(p')'=m+\omega m-f$ in the sense of distributions, implying that $p'\in H^1(0,L)$. Furthermore, after one integration by parts in (3.8) we get

$$\int_0^L d\left(m-m''+\omega m-f\right)dx=0\quad\forall d\in C_0^\infty(0,L),$$

which tells us that $m-m''+\omega m=f$ for almost every $x\in(0,L)$. Using this conclusion in the expression obtained after one integration by parts in (3.8), we arrive at

$$-m'(0)d(0)+\left[m'(L)+\alpha_\lambda(m(L))\right]d(L)=0\quad\forall d\in H^1(0,L).$$

Accordingly, so far we have shown:

Lemma 3.1. *For any $\lambda\in(0,\infty)$ there exists $p_\lambda\in H^2(0,L)$ such that $p_\lambda-p_\lambda''+\omega p_\lambda=f$ for almost every $x\in(0,L)$, $p_\lambda'(0)=0$ and $p_\lambda'(L)+\alpha_\lambda(p_\lambda(L))=0$.*

Let us consider the p_λ given by Lemma 3.1. We proceed to prove that $R(I+\mathcal{A})=L^2(0,L)$ by analyzing what happens to p_λ as $\lambda\to 0^+$. To this end, we shall obtain upper bounds for $\|p_\lambda\|_{H^2(0,L)}$ and $|\alpha_\lambda(p_\lambda(L))|$, both independent of $\lambda\in(0,\infty)$.

- Upper bound for $\|p_\lambda\|_{H^2(0,L)}$ independent of $\lambda\in(0,\infty)$.
  In virtue of Lemma 3.1 we have
  $$\left(\frac{1}{2}+\omega\right)\int_0^L|p_\lambda|^2\,dx+\int_0^L|p_\lambda'|^2\,dx-p_\lambda'(L)p_\lambda(L)\leq\frac{1}{2}\int_0^L|f|^2\,dx.\tag{3.9}$$
  Since $0\in\alpha(0)$ we get $J_\lambda(0)=0$, and consequently, $\alpha_\lambda(0)=0$. Then, as α_λ is a (maximal) monotone operator we infer
  $$-p_\lambda'(L)p_\lambda(L)=\alpha_\lambda(p_\lambda(L))p_\lambda(L)=\left[\alpha_\lambda(p_\lambda(L))-\alpha_\lambda(0)\right]\left(p_\lambda(L)-0\right)\geq 0,$$
  which considered in (3.9) allows us to obtain
  $$\|p_\lambda\|_{H^1(0,L)}\leq\|f\|_{L^2(0,L)}.\tag{3.10}$$
  Moreover, since $p_\lambda-p_\lambda''+\omega p_\lambda=f$ for almost every $x\in(0,L)$ we actually have
  $$\|p_\lambda''\|_{L^2(0,L)}\leq(2+\omega)\|f\|_{L^2(0,L)}.\tag{3.11}$$
- Upper bound for $|\alpha_\lambda(p_\lambda(L))|$ independent of $\lambda\in(0,\infty)$.
  The injection of $H^1(0,L)$ into $C([0,L])$ is continuous, and hence, there exists a constant $C_I\in(0,\infty)$ such that $\|p\|_{C([0,L])}\leq C_I\|p\|_{H^1(0,L)}$ for any $p\in H^1(0,L)$. Then, by Lemma 3.1 we obtain
  $$|\alpha_\lambda(p_\lambda(L))|=|p_\lambda'(L)|\leq\|p_\lambda'\|_{C([0,L])}\leq C_I\|p_\lambda'\|_{H^1(0,L)}.\tag{3.12}$$

Therefore, in view of (3.10), (3.11) and (3.12) we see that the sequences $\left(p_\lambda\right)_{\lambda\geq0}\subset H^2(0,L)$ and $\left(\alpha_\lambda(p_\lambda(L))\right)_{\lambda\geq0}\subset\mathbb{R}$ are bounded, and hence, there exist $(p,g)\in H^2(0,L)\times\mathbb{R}$ and subsequences, which we denote by the same symbols, such that $p_\lambda\rightharpoonup p$ in $H^2(0,L)$ (weak convergence) and $\alpha_\lambda(p_\lambda(L))\to g$ in $\mathbb{R}$ as $\lambda\to 0^+$. Moreover, we infer that $p_\lambda\to p$ in $C^1([0,L])$ as $\lambda\to 0^+$ since the injection of $H^2(0,L)$ into $C^1([0,L])$ is compact, implying that $p_\lambda'(x)\to p'(x)$ in $\mathbb{R}$ as $\lambda\to 0^+$ for all $x\in[0,L]$.

Accordingly, Lemma 3.1 and the previous arguments yield:

Lemma 3.2. *There exists $p\in H^2(0,L)$ such that $p-p''+\omega p=f$ for almost every $x\in(0,L)$, $p'(0)=0$ and $p'(L)+g=0$.*

In view of Lemma 3.2 and (3.3) we see that in order to complete the proof of $R(I+\mathcal{A})=L^2(0,L)$ we just need to prove that $g\in\alpha(p(L))=D\text{sign}(p(L))$. The required arguments can be found at the end of the proof of [5, Proposition 3.1], but we present it again here for the sake of completeness. We make use of several results that come from the fact that α is a maximal monotone operator. Let us recall that $p_\lambda(L) \to p(L)$ in $\mathbb{R}$ and $\alpha_\lambda(p_\lambda(L)) \to g$ in $\mathbb{R}$ as $\lambda \to 0^+$ (the subsequences). Then, since the Yosida approximation of α satisfies that $\alpha_\lambda(x) \in \alpha\left(J_\lambda(x)\right)$ for all $x \in \mathbb{R}$ and α is a closed operator, it suffices to prove that $J_\lambda(p_\lambda(L)) \to p(L)$ in $\mathbb{R}$ as $\lambda \to 0^+$ to deduce that $g \in \alpha(p(L))$. Taking into account that $|J_\lambda(x_1) - J_\lambda(x_2)| \le |x_1 - x_2|$ for all $(x_1, x_2) \in \mathbb{R}^2$ it follows

$$\begin{aligned} &|J_\lambda(p_\lambda(L)) - p(L)| \\ &= |J_\lambda(p_\lambda(L)) - J_\lambda(p(L)) + J_\lambda(p(L)) - p(L)| \\ &\le |p_\lambda(L) - p(L)| + |J_\lambda(p(L)) - p(L)|. \end{aligned}$$

Finally, we deduce that $J_\lambda(p_\lambda(L)) \to p(L)$ in $\mathbb{R}$ as $\lambda \to 0^+$ because the resolvent of α satisfies that $J_\lambda(x) \to x$ in $\mathbb{R}$ as $\lambda \to 0^+$ for all $x \in \mathbb{R}$. The proof of Proposition 3.2 is complete. ■We proceed to show the well-posedness part of Theorem 1.1.

In view of Propositions 3.1, 3.2 and [21, Chapter IV, Lemma 1.3] we see that the operator $\mathcal{A}$, defined by (3.3), is maximal monotone.

Let us recall that

$$f(t,x) = -d'(t)\phi(x) + d(t)\phi''(x) - \omega d(t)\phi(x),$$

$$w_0(x) = y_0(x),$$

$$y_0(x) = z_0(x) + \int_0^x k(x,s) z_0(s)\, ds,$$

which by hypotheses satisfy $f \in W^{1,1}(0,\infty; L^2(0,L))$ and $w_0 \in D(\mathcal{A})$. Then, regarding the well-posedness of (3.4), we have that [21, Chapter IV, Theorem 4.1] tells us the existence of a unique $w \in W^{1,1}(0,\infty; L^2(0,L))$ such that $w'(t) + \mathcal{A}w(t) = f(t)$ for almost every $t > 0$, $w(t) \in D(\mathcal{A})$ for every $t \ge 0$ and $w(0) = w_0$. Accordingly, there exists a unique $w = w(t,x)$ in $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ such that

$$\begin{cases} w_t - w_{xx} + \omega w = f \text{ for almost every } (t,x) \in (0,\infty) \times (0,L), \\ w_x(t,0) = 0 \text{ for every } t \in [0,\infty), \\ w_x(t,L) + D\mathrm{sign}(w(t,L)) \ni 0 \text{ for every } t \in [0,\infty), \\ w(0,x) = w_0(x) \text{ for every } x \in (0,L). \end{cases} \tag{3.13}$$

Then, as $w(t,x) = y(t,x) - d(t)\phi(x)$ and both the backstepping transformation and its inverse are linear and bounded operators, results given by [1, Lemma 3.3], we obtain the existence of a unique $z = z(t,x)$ in $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ solving (1.8).

The proof of Theorem 1.1 is complete.

4. Concluding remarks

In this paper we have shown the rapid stabilization (1.3) of the unstable heat Eq. (1.1), in which there is an unknown disturbance acting at the flux boundary condition. We have employed the backstepping method, Lyapunov techniques and the sign multivalued operator (1.6) for the design of the multivalued feedback law (1.7). The assumptions made on the unknown disturbance, namely **(A1)** and **(A2)**, are the standard ones that can be found in the literature. The well-posedness of the corresponding closed-loop system (1.8) has been shown with the maximal monotone operator theory.

The main contribution of this paper, besides the rapid stabilization result (Remark 1.1), is the idea behind the feedback design: to split the control into two parts (Remark 2.1), with one part for the employing of the backstepping method, used to achieve the exponential decay of the energy (1.2) for any desired decay rate, and the other part for the handling of the unknown disturbance, in which it is used the sign multivalued operator (1.6). We think that this approach could be applied to address the rapid stabilization of other partial differential equations subjected to disturbances.

In this paper we have considered [1] as a basis. Accordingly, we believe that a natural open problem would be to obtain the rapid stabilization of the unstable wave equation considered in [26], but with an unknown disturbance acting at the boundary condition.

CRediT authorship contribution statement

Patricio Guzmán: Writing – review & editing, Writing – original draft, Supervision, Investigation, Conceptualization. **Esteban Hernández:** Writing – review & editing, Writing – original draft, Supervision, Investigation, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Patricio Guzman reports financial support was provided by National Agency for Research and Development. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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