

1 Energy decay of a microbeam model
2 with a locally distributed nonlinear feedback control

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4 **Abstract**

5 In this paper we address the problem of internal stabilization of the deflection of a microbeam, which
6 is modeled by a sixth-order hyperbolic equation. Employing multiplier techniques and an integral
7 inequality, we prove that a locally distributed nonlinear feedback control forces the energy associated
8 to the deflection to decay exponentially or polynomially to zero. As a consequence of this, the deflection
9 goes to the rest position as the time goes to infinity.

10 **2010 Mathematics Subject Classification:** 35B40, 35L35, 74K10, 93B52.

11 **Keywords:** Microbeam model, hyperbolic equation, internal stabilization, locally distributed
12 nonlinear feedback control, exponential energy decay, polynomial energy decay.

13 **1 Introduction**

14 A microbeam is a beam whose dimensions are in the order of a few microns. According to
15 [26, Chapter 6], the microbeams are perhaps the most common structural component used in micro-
16 electro-mechanical systems (MEMS) such as actuators, filters, resonators and sensors.

17
18 The deflection $z = z(t, x)$ of a clamped microbeam of density $\rho > 0$, cross-sectional area $A > 0$,
19 Young's modulus $E > 0$, area moment of inertia $I > 0$, shear modulus $G > 0$ and length $L > 0$ being
20 subjected to a distributed load $f = f(t, x)$ can be modeled by

$$\left\{ \begin{array}{l} \rho A z_{tt} + M_1 z_{xxxx} - M_2 z_{xxxxx} = f, \quad (t, x) \in (0, \infty) \times (0, L), \\ z(t, 0) = z_x(t, 0) = z_{xx}(t, 0) = 0, \quad t \in (0, \infty), \\ z(t, L) = z_x(t, L) = z_{xx}(t, L) = 0, \quad t \in (0, \infty), \\ z(0, x) = z_0(x), \quad x \in (0, L), \\ z_t(0, x) = z_1(x), \quad x \in (0, L). \end{array} \right. \quad (1.1)$$

21 This model has been derived in [12, Section 3] and [17, Section 4] by using the modified strain gradient
22 elasticity theory developed in [19, Section 2] together with Hamilton's Principle. Here

$$M_1 = EI + GA \left(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2 \right) \quad \text{and} \quad M_2 = GA \left(2l_0^2 + \frac{4}{5}l_1^2 \right),$$

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1 where $l_0 > 0$, $l_1 > 0$ and $l_2 > 0$ are the material length scale parameters associated with dilatation
 2 gradients, deviatoric stretch gradients and rotation gradients, respectively. These parameters were
 3 introduced in [19, Section 2] to characterize the phenomenon observed in experiments that the
 4 deformation of some materials is size-dependent. Further information on microbeams, its related
 5 experiments and also on MEMS may be found in [12, 17, 19, 26] and the references therein.

6
 7 Without loss of generality, and in order to ease the notation throughout this paper, from now on
 8 instead of considering (1.1) we consider

$$\left\{ \begin{array}{l} z_{tt} + z_{xxxx} - z_{xxxxx} = f, \quad (t, x) \in (0, \infty) \times (0, 1), \\ z(t, 0) = z_x(t, 0) = z_{xx}(t, 0) = 0, \quad t \in (0, \infty), \\ z(t, 1) = z_x(t, 1) = z_{xx}(t, 1) = 0, \quad t \in (0, \infty), \\ z(0, x) = z_0(x), \quad x \in (0, 1), \\ z_t(0, x) = z_1(x), \quad x \in (0, 1). \end{array} \right. \quad (1.2)$$

9 In this paper we address the problem of internal stabilization of (1.2). Let us present such a
 10 problem. For a regular enough solution $z = z(t, x)$ of (1.2) we define its energy by

$$E(t) = \frac{1}{2} \int_0^1 (|z_t|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx, \quad t \in [0, \infty). \quad (1.3)$$

11 Then, formal computations yield

$$E'(t) = \int_0^1 f z_t dx, \quad t \in [0, \infty), \quad (1.4)$$

12 from which it follows that the energy is conserved when $f = f(t, x)$ is not present. With the purpose
 13 of stabilizing $z = z(t, x)$ to the rest position, that is to say, $\lim_{t \rightarrow \infty} z(t, \cdot) = 0$ and $\lim_{t \rightarrow \infty} z_t(t, \cdot) = 0$,
 14 we take a $f = f(t, x)$ that forces the dissipation of the energy, and moreover, its decay.

15
 16 From (1.4) we see that we could choose

$$\left. \begin{array}{l} f(t, x) = f(z_t(t, x)) = -a(x)z_t(t, x) \quad \text{with} \\ a \in L^\infty(0, 1) \quad \text{such that} \quad a(x) \geq a_0 > 0 \quad \text{for almost every} \quad x \in (0, 1). \end{array} \right\} \quad (1.5)$$

17 With the choice of (1.5), which is a linear feedback control distributed over the whole domain, the
 18 dissipation of the energy is ensured. Moreover, the exponential decay of the energy can be obtained
 19 by applying the ideas introduced by Haraux and Zuazua in [11] to

$$E_\varepsilon(t) = E(t) + \varepsilon \int_0^1 z z_t dx, \quad t \in [0, \infty), \quad (1.6)$$

20 where $\varepsilon > 0$ is a constant that must be suitably chosen. Let us note that (1.6) can be seen as a
 21 perturbation of (1.3). Those ideas were also applied in [3, 6, 7, 16, 21, 27, 28] to obtain the exponential
 22 or polynomial decay of the energy associated to the solution of some hyperbolic equations with linear
 23 or nonlinear feedback controls acting either in the domain or at the boundary.

1 Even if (1.5) solves our stabilization problem, acting over the whole domain might be too
2 restrictive. For this reason instead of choosing (1.5) we choose a nonlinear feedback control locally
3 distributed over the domain, for being less restrictive and more general. In order to present it, we need
4 to make some assumptions.

5 (A1) (Localization) Let $\omega \subset (0, 1)$ be a non-empty interval. Let $a \in L^\infty(0, 1)$ be a non-negative
6 function such that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$.

7 (A2) (Nonlinearity) Let $g \in C(\mathbb{R})$ be a non-decreasing function satisfying that there exist constants
8 $R \in [1, \infty)$, $S \in [1, \infty)$ and $(n_1, n_2, n_3, n_4) \in (0, \infty)^4$ such that $n_1|x|^R \leq |g(x)| \leq n_2|x|^{\frac{1}{R}}$ if $|x| \leq 1$
9 and $n_3|x| \leq |g(x)| \leq n_4|x|^S$ if $|x| > 1$.

10 Then, with the previous assumptions in mind we choose

$$f(t, x) = f(z_t(t, x)) = -a(x)g(z_t(t, x)). \quad (1.7)$$

11 **Remark 1.1** In (A1) we could take $a(x) = \mathbb{1}_\omega(x)$, where $\mathbb{1}_\omega$ denotes the characteristic function on ω ,
12 that is to say, $\mathbb{1}_\omega(x) = 1$ if $x \in \omega$ and $\mathbb{1}_\omega(x) = 0$ if $x \notin \omega$. The assumption that $a(x) \geq a_0 > 0$ for
13 almost every $x \in \omega$ in (A1) means that (1.7) acts only through ω .

14 **Remark 1.2** In (A2) we could take $g(x) = |x|^{R-1}x$ for any $R \in [1, \infty)$. The type of nonlinearity in
15 (A2) is similar to the ones in [13, 22], the same as the ones in [8, 15, 18] and more general than the
16 ones in [3, 9, 21, 27, 28]. Finally, from (A2) we deduce

$$(g(x) - g(y))(x - y) \geq 0 \text{ for every } (x, y) \in \mathbb{R}^2 \text{ and } g(0) = 0. \quad (1.8)$$

17 In virtue of (1.4) and (1.7) we have

$$E'(t) = - \int_0^1 ag(z_t)z_t dx, \quad t \in [0, \infty), \quad (1.9)$$

18 from which it follows that the energy still dissipates due to (A1) and (1.8). Therefore, it is natural to
19 wonder if the energy still decays. Our main result gives a positive answer to this question, thus also
20 giving an answer to our stabilization problem.

21 **Theorem 1.1** Let us assume (A1) and (A2). Let $(z_0, z_1) \in H_0^3(0, 1) \times L^2(0, 1)$. Then, (1.2) with (1.7)
22 has a unique weak solution $z \in C([0, \infty); H_0^3(0, 1)) \cap C^1([0, \infty); L^2(0, 1))$. Moreover, its energy decays
23 exponentially or polynomially to zero. Being precise:

24

25 • (Exponential decay) If $R = 1$, then there exists a constant $C_1 > 0$ such that

$$E(t) \leq E(0) \exp \left\{ 1 - \frac{t}{C_1 \left(1 + E(0)^{\frac{S-1}{2S}} \right)} \right\}, \quad t \in [0, \infty). \quad (1.10)$$

26 • (Polynomial decay) If $R > 1$, then there exists a constant $C_2 > 0$ such that

$$E(t) \leq C_2 \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right)^{\frac{2}{R-1}} t^{-\frac{2}{R-1}}, \quad t \in (0, \infty). \quad (1.11)$$

Let us make some comments on the proof of this theorem. Its well-posedness part is shown with the aid of the semigroup theory, while its energy decay part is shown by adapting a method introduced by Tebou in [22] to address the problem of internal stabilization of the wave equation. We have followed this approach because we have not been able to find a perturbation of (1.3), like the one given in (1.6), to prove (1.10) and (1.11).

The method introduced by Tebou in [22] allows to obtain, by employing multiplier techniques and an integral inequality derived by Komornik in [13], the exponential or polynomial decay of the energy associated to the solution of a large class of hyperbolic equations with locally distributed nonlinear feedback controls. However, that method to work requires the nonlinear feedback control to be localized on a suitable neighborhood of the boundary. Then, we see from [22, Pages 502 and 503] that we should have assumed (A0) instead of (A1), where:

(A0) (Localization) Let $\varepsilon \in (0, 1)$. Let us consider either $\omega = (0, \varepsilon)$ or $\omega = (1 - \varepsilon, 1)$. Let $a \in L^\infty(0, 1)$ be a non-negative function such that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$.

One of the contributions of this paper is the adaptation of the method introduced by Tebou in [22] to the one-dimensional case in such a way that (1.10) and (1.11) can still be shown with a weaker assumption than (A0), namely (A1).

Regarding the study of the control properties of microbeams, we mention that the exact boundary controllability was analyzed in [10, 24], the boundary stabilization with linear feedback controls was addressed in [23, 25] and the exact boundary observability was studied in [5].

The rest of this paper is organized as follows. In Section 2 we prove the well-posedness results needed for the study of (1.2) with (1.7). Then, in Section 3 we prove our main result, which is the exponential or polynomial decay of the energy given by Theorem 1.1.

2 Well-posedness

In this section we apply the semigroup theory exposed in [1] to obtain the well-posedness results needed for the study of (1.2) with (1.7). We proceed as in [8, 9, 14]. Let us introduce some notations. Let us consider the Hilbert spaces $H = L^2(0, 1)$ and $V = H_0^3(0, 1)$ respectively equipped with the inner products

$$(z, y)_H = \int_0^1 zy \, dx \quad \text{and} \quad (z, y)_V = \int_0^1 (z''y'' + z'''y''') \, dx.$$

Note that the latter can be done due to Poincaré's inequality. We identify H with itself to obtain the continuous and dense injections $V \hookrightarrow H \hookrightarrow V^*$, where $V^* = H^{-3}(0, 1)$ is the dual space of V . We employ $\langle \cdot, \cdot \rangle_{V^* \times V}$ to denote the duality product between V^* and V . Let us define the operators $A : V \rightarrow V^*$ and $B : V \rightarrow H$ respectively through

$$\langle Az, y \rangle_{V^* \times V} = (z, y)_V \quad \text{and} \quad Bz = ag(z). \tag{2.1}$$

1 **Lemma 2.1** *The operators defined through (2.1) are well defined.*

2 **Proof.** The operator $A : V \rightarrow V^*$ is defined through a bilinear continuous form, thus it is well defined.
 3 Introducing the sets $\Omega_1 = \{x \in (0, 1) / |z(x)| \leq 1\}$ and $\Omega_2 = \{x \in (0, 1) / |z(x)| > 1\}$ we see that
 4 the operator $B : V \rightarrow H$ is also well defined. Indeed, thanks to (A2), Hölder's inequality and the
 5 continuous injection $V \hookrightarrow L^\infty(0, 1)$ there exists a constant $C > 0$ such that

$$\begin{aligned} \|Bz\|_H^2 &\leq \|a\|_{L^\infty(\Omega_1)}^2 n_2^2 \int_{\Omega_1} |z|^{\frac{2}{h}} dx + \|a\|_{L^\infty(\Omega_2)}^2 n_4^2 \|z\|_{L^\infty(\Omega_2)}^{2S-2} \int_{\Omega_2} |z|^2 dx \\ &\leq C \left(\int_{\Omega_1} |z|^2 dx \right)^{\frac{1}{h}} + C \|z\|_V^{2S-2} \int_{\Omega_2} |z|^2 dx, \quad z \in V. \end{aligned} \quad (2.2)$$

6 The proof of Lemma 2.1 is complete. ■

7

8 Additionally, we also consider the Hilbert space $\mathcal{H} = V \times H$ equipped with the inner product

$$((z, y), (Z, Y))_{\mathcal{H}} = (z, Z)_V + (y, Y)_H.$$

9 Finally, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ by

$$D(\mathcal{A}) = \{(z, y) \in V \times V / Az + By \in H\} \quad \text{and} \quad \mathcal{A}(z, y) = (-y, Az + By). \quad (2.3)$$

10 **Lemma 2.2** *We have that $D(\mathcal{A}) = (H^6(0, 1) \cap V) \times V$.*

11 **Proof.** Let us show that $(H^6(0, 1) \cap V) \times V \subset D(\mathcal{A})$. By Lemma 2.1 we only need to show that
 12 $z \in H^6(0, 1) \cap V$ implies that $Az \in H$. Assuming that $z \in H^6(0, 1) \cap V$, three integrations by parts
 13 and the fact that $z'''' - z'''''' \in H$ allow us to deduce the desired result since

$$\langle Az, \phi \rangle_{V^* \times V} = (z, \phi)_V = (z'''' - z''''', \phi)_H = \langle z'''' - z''''', \phi \rangle_{V^* \times V}, \quad \phi \in V.$$

14 Now, let us show that $D(\mathcal{A}) \subset (H^6(0, 1) \cap V) \times V$. By Lemma 2.1 we only need to show that $z \in V$
 15 and $Az \in H$ imply that $z \in H^6(0, 1) \cap V$. Assuming that $z \in V$ and $Az \in H$, one integration by parts
 16 yields

$$(Az, \phi)_H = \langle Az, \phi \rangle_{V^* \times V} = (z, \phi)_V = (-1)^3 (z' - z''', \phi''')_H, \quad \phi \in C_0^\infty(0, 1),$$

17 from which it follows that $Az = (z' - z''')'''$ in the sense of distributions. Finally, taking into account
 18 [2, Chapter 8, Remark 7], we deduce the desired result since $z' - z''' \in H$ and $(z' - z''')''' \in H$. The
 19 proof of Lemma 2.2 is complete. ■

20

21 Let us note that in the proof of Lemma 2.2 we have deduced that $Az = z'''' - z''''''$ if
 22 $z \in H^6(0, 1) \cap V$. Then, with the aid of the operator defined by (2.3) and Lemma 2.2 we see that (1.2)
 23 with (1.7) can be written as

$$\left. \begin{aligned} \frac{d}{dt} (z, z_t) + \mathcal{A}(z, z_t) &= 0, \quad t \in [0, \infty), \\ (z, z_t)(0) &= (z_0, z_1). \end{aligned} \right\} \quad (2.4)$$

1 We are going to obtain some well-posedness results for (1.2) with (1.7) by applying the semigroup
2 theory to (2.4). The main result of this section is the following one.

3 **Proposition 2.1** *The operator defined by (2.3) is a maximal monotone operator in \mathcal{H} .*

4 **Proof.** In view of [1, Proposition 2.2] we see that it suffices to prove that the operator \mathcal{A} is monotone
5 and that $R(I + \mathcal{A}) = \mathcal{H}$. Let us prove the first part. Let (z, y) and (Z, Y) be elements of $D(\mathcal{A})$. Then,
6 thanks to Lemma 2.2, three integrations by parts, (A1) and (1.8) we have

$$\begin{aligned} & (\mathcal{A}(z, y) - \mathcal{A}(Z, Y), (z, y) - (Z, Y))_{\mathcal{H}} \\ &= ((-(y - Y), A(z - Z)), (z - Z, y - Y))_{\mathcal{H}} + ((0, By - BY), (z - Z, y - Y))_{\mathcal{H}} \\ &= -(y - Y, z - Z)_V + (A(z - Z), y - Y)_H + (By - BY, y - Y)_H \\ &= (By - BY, y - Y)_H \geq 0. \end{aligned}$$

7 Now, let us prove the second part, that is to say, to prove that for any $(z_0, y_0) \in \mathcal{H}$ there exists
8 $(z, y) \in D(\mathcal{A})$ such that $(I + \mathcal{A})(z, y) = (z - y, y + Az + By) = (z_0, y_0)$. To this end, for $y_0 - Az_0 \in V^*$
9 let us consider the problem:

$$\text{Find } y \in V \text{ such that } y + Ay + By = y_0 - Az_0 \text{ in } V^*. \quad (2.5)$$

10 Let us assume that we have solved (2.5). Then, introducing $z = y + z_0 \in V$ we see from (2.5) that
11 $Az + By = y_0 - y$ in V^* , which tells us that $Az + By \in H$ because $y_0 - y \in H$. Accordingly, we
12 have found a $(z, y) \in D(\mathcal{A})$ such that $(I + \mathcal{A})(z, y) = (z - y, y + Az + By) = (z_0, y_0)$. Now, it only left
13 to solve (2.5). In order to do so, let us define the functional $J : V \rightarrow \mathbb{R}$ by

$$J(y) = \frac{1}{2} \|y\|_H^2 + \frac{1}{2} \|y\|_V^2 + \int_0^1 G(y) dx - \langle y_0 - Az_0, y \rangle_{V^* \times V} \quad \text{with } G(s) = a \int_0^s g(t) dt.$$

14 Since (A1) and (1.8) imply

$$0 \leq G(y) \leq (By)y, \quad y \in V, \quad (2.6)$$

15 it follows that the functional is well defined due to the Cauchy-Schwarz inequality and (2.2). Indeed,
16 there exists a constant $C > 0$ such that

$$|J(y)| \leq \frac{1}{2} \|y\|_H^2 + \frac{1}{2} \|y\|_V^2 + C \left(\|y\|_H^{\frac{1}{R}} + \|y\|_V^{S-1} \|y\|_H \right) \|y\|_H + \|y_0 - Az_0\|_{V^*} \|y\|_V, \quad y \in V.$$

17 Now, let us show that the functional is convex, coercive and lower-semicontinuous. In view of (A1)
18 and (1.8) we get

$$(G'(y_1) - G'(y_2))(y_1 - y_2) \geq 0, \quad (y_1, y_2) \in V \times V,$$

1 which allows us to conclude that the functional is convex (see [20, Proposition 3.10]). From (2.6) and
 2 the Cauchy–Schwarz inequality we get

$$J(y) \geq \frac{1}{2} \|y\|_V^2 - \|y_0 - Az_0\|_{V^*} \|y\|_V, \quad y \in V,$$

3 thus implying that the functional is coercive because $J(y) \rightarrow \infty$ as $\|y\|_V \rightarrow \infty$. Finally, the functional
 4 is lower–semicontinuous because it is continuous. Indeed, by (1.8), (2.2) and the Cauchy–Schwarz
 5 inequality there exists a constant $C > 0$ such that

$$\begin{aligned} \|G(y_1) - G(y_2)\|_{L^1(0,1)} &\leq \|By_2\|_H \|y_1 - y_2\|_H \\ &\leq C \left(\|y_2\|_H^{\frac{1}{R}} + \|y_2\|_V^{S-1} \|y_2\|_H \right) \|y_1 - y_2\|_H, \quad (y_1, y_2) \in V \times V. \end{aligned}$$

6 Therefore, we infer the existence of a $\hat{y} \in V$ minimizing the functional (see [20, Theorem 2.19]). Since
 7 such a minimizer $\hat{y} \in V$ satisfies

$$0 = \langle J'(\hat{y}), \phi \rangle_{V^* \times V} = \langle \hat{y} + A\hat{y} + B\hat{y} - (y_0 - Az_0), \phi \rangle_{V^* \times V}, \quad \phi \in V,$$

8 we see that $\hat{y} \in V$ solves (2.5). The proof of Proposition 2.1 is complete. ■

9
 10 Let us apply the semigroup theory to (2.4). For every $(z_0, z_1) \in D(\mathcal{A})$ we have that (2.4) has
 11 a unique strong solution $(z, z_t) \in L^\infty(0, \infty; D(\mathcal{A})) \cap W^{1,\infty}(0, \infty; \mathcal{H})$ in virtue of Proposition 2.1 and
 12 [1, Theorem 3.1]. Furthermore, its solution map $(z_0, z_1) \in D(\mathcal{A}) \subset \mathcal{H} \mapsto (z, z_t)(t, \cdot) \in \mathcal{H}$ is a
 13 contraction for each $t \in [0, \infty)$. Since $D(\mathcal{A})$ is dense in \mathcal{H} due to Lemma 2.2, such a solution map can
 14 be extended by continuity to a unique map $S(t) : \mathcal{H} \rightarrow \mathcal{H}$ so that $(z, z_t)(t, \cdot) = S(t)(z_0, z_1)$ for each
 15 $t \in [0, \infty)$ when $(z_0, z_1) \in D(\mathcal{A})$. It can be checked that $(S(t))_{t \geq 0}$ is a strongly continuous semigroup of
 16 contractions in \mathcal{H} (see [1, Page 55]), thus for any $(z_0, z_1) \in \mathcal{H}$ we may define the unique weak solution
 17 of (2.4) by $S(t)(z_0, z_1)$ with the property that $t \in [0, \infty) \mapsto S(t)(z_0, z_1) \in \mathcal{H}$ is a continuous map.

18 **Remark 2.1** For a given $(z_0, z_1) \in \mathcal{H}$ let us take the unique weak solution $S(t)(z_0, z_1)$ of (2.4).
 19 Since $D(\mathcal{A})$ is dense in \mathcal{H} due to Lemma 2.2, there exists a sequence $\{(z_{0n}, z_{1n})\}_{n \in \mathbb{N}} \subset D(\mathcal{A})$ such
 20 that $(z_{0n}, z_{1n}) \rightarrow (z_0, z_1)$ in \mathcal{H} as $n \rightarrow \infty$. Now, for each $n \in \mathbb{N}$ let us take the corresponding
 21 unique strong solution $S(t)(z_{0n}, z_{1n})$ of (2.4). Therefore, since $(S(t))_{t \geq 0}$ is a strongly continuous
 22 semigroup of contractions in \mathcal{H} , for each $t \in [0, \infty)$ we have that $S(t)(z_{0n}, z_{1n}) \rightarrow S(t)(z_0, z_1)$ in \mathcal{H} as
 23 $n \rightarrow \infty$.

24 The previous analysis is translated into the following result.

25 **Proposition 2.2** Let us assume (A1) and (A2).

26 (i) Let $(z_0, z_1) \in (H^6(0, 1) \cap V) \times V$. Then, (1.2) with (1.7) has a unique strong solution $z \in$
 27 $L^\infty(0, \infty; H^6(0, 1) \cap V) \cap W^{1,\infty}(0, \infty; V) \cap W^{2,\infty}(0, \infty; H)$.

1 (ii) Let $(z_0, z_1) \in V \times H$. Then, (1.2) with (1.7) has a unique weak solution $z \in C([0, \infty); V) \cap$
 2 $C^1([0, \infty); H)$.

3 Internal stabilization

4 In this section we prove our main result, which is the exponential or polynomial decay of the
 5 energy given by Theorem 1.1. To this end, we are going to apply an integral inequality derived by
 6 Komornik that can be found in [13, Theorem 1.4] or [14, Theorem 9.1].

7 **Theorem 3.1** Let $E : [0, \infty) \rightarrow [0, \infty)$ be a non-increasing function. Assume that there exist constants
 8 $\alpha \geq 0$ and $\beta > 0$ such that

$$\int_t^\infty E(\tau)^{\alpha+1} d\tau \leq \beta E(t), \quad t \in [0, \infty). \quad (3.1)$$

9 Then, we have

$$E(t) \leq \begin{cases} E(0) \exp\left\{1 - \frac{t}{\beta}\right\}, & t \in [0, \infty), \quad \text{if } \alpha = 0, \\ \beta^{\frac{1}{\alpha}} \left(1 + \frac{1}{\alpha}\right)^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}}, & t \in (0, \infty), \quad \text{if } \alpha > 0. \end{cases} \quad (3.2)$$

10 **Remark 3.1** The second line of the right-hand side of (3.2), corresponding to the case $\alpha > 0$, is a
 11 consequence of [13, Section 1, (1.22)] or [14, Section 9, (2)]. We have preferred the current expression
 12 to clearly appreciate the polynomial behavior.

13 **Proof of Theorem 1.1.** In virtue of the density argument presented in Remark 2.1, it suffices to
 14 perform all the required computations assuming that $(z_0, z_1) \in (H^6(0, 1) \cap H_0^3(0, 1)) \times H_0^3(0, 1)$, so that
 15 (1.2) with (1.7) has by Proposition 2.2 a unique strong solution $z \in L^\infty(0, \infty; H^6(0, 1) \cap H_0^3(0, 1)) \cap$
 16 $W^{1, \infty}(0, \infty; H_0^3(0, 1)) \cap W^{2, \infty}(0, \infty; L^2(0, 1))$.

17
 18 In view of (A1) and (1.8) we see that multiplying (1.2) with (1.7) by $z_t(t, x)$ and then
 19 performing some integrations by parts on $(0, 1)$ yield

$$E'(t) = - \int_0^1 ag(z_t)z_t dx \leq 0, \quad t \in [0, \infty), \quad (3.3)$$

20 which tells us that the energy is a non-increasing function. Then, in order to apply Theorem 3.1 we
 21 need to obtain constants $\alpha \geq 0$ and $\beta > 0$ such that the energy satisfies (3.1). This will be done by
 22 employing several multiplier techniques. We split the proof into four parts.

23
 24 **Part 1.** Let $0 \leq t_1 < t_2 < \infty$. Let $m \in C^\infty([0, 1])$ be such that $m(0) = 0$. Let $\alpha \geq 0$ to
 25 be determined later. Multiplying (1.2) with (1.7) by $E(t)^\alpha m(x)z_x(t, x)$ and then performing some
 26 integrations by parts on $(t_1, t_2) \times (0, 1)$ we obtain

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_0^1 \frac{1}{2} E^\alpha m' (|z_t|^2 + 3|z_{xx}|^2 + 5|z_{xxx}|^2) dx dt + \int_0^1 E^\alpha m z_x z_t \Big|_{t_1}^{t_2} dx \\
& - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' m z_x z_t dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha m'' (z_{xx} z_x + 2z_{xxx} z_{xx}) dx dt \\
& + \int_{t_1}^{t_2} \int_0^1 E^\alpha (m''' z_x + m'' z_{xx}) z_{xxx} dx dt \\
& + \int_{t_1}^{t_2} \int_0^1 E^\alpha m z_x ag(z_t) dx dt = \int_{t_1}^{t_2} \frac{1}{2} E^\alpha m(1) |z_{xxx}(t, 1)|^2 dt. \quad (3.4)
\end{aligned}$$

1 Then, choosing $m(x) = x$ in (3.4) leads us to

$$\begin{aligned}
& \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt + \int_0^1 E^\alpha x z_x z_t \Big|_{t_1}^{t_2} dx \\
& - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' x z_x z_t dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha x z_x ag(z_t) dx dt = \int_{t_1}^{t_2} \frac{1}{2} E^\alpha |z_{xxx}(t, 1)|^2 dt. \quad (3.5)
\end{aligned}$$

2 For a fixed $x_0 \in \text{int}(\omega)$ let us introduce $\varepsilon_0 = \text{dist}(x_0, \partial\omega)/2 > 0$ and $\omega_0 = (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$,
3 thus obtaining that $\overline{\omega_0} \subset \omega$. Then, let us take a function $\chi_0 \in C^\infty([0, 1])$ satisfying that $\chi_0 = 0$ in
4 $[0, x_0 - \varepsilon_0]$ and $\chi_0 = 1$ in $[x_0 + \varepsilon_0, 1]$. This time we see that choosing $m(x) = \chi_0(x)$ in (3.4) and then
5 using Cauchy's inequality we get a constant $C_1 > 0$ such that

$$\begin{aligned}
& \int_{t_1}^{t_2} \frac{1}{2} E^\alpha |z_{xxx}(t, 1)|^2 dt \leq \int_0^1 E^\alpha \chi_0 z_x z_t \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_0 z_x z_t dx dt \\
& + \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_0 z_x ag(z_t) dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt \\
& + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt,
\end{aligned}$$

6 and hence, from (3.5) it follows

$$\begin{aligned}
& \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt \leq \int_0^1 E^\alpha (-x + \chi_0) z_x z_t \Big|_{t_1}^{t_2} dx \\
& + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' (x - \chi_0) z_x z_t dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (-x + \chi_0) z_x ag(z_t) dx dt \\
& + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt. \quad (3.6)
\end{aligned}$$

7 Let us handle the first two terms of the right-hand side of (3.6). Using that (3.3) implies that
8 $E(t_2) \leq E(t_1) \leq E(0)$ whenever $0 \leq t_1 < t_2 < \infty$, the Cauchy and Poincaré inequalities give us a
9 constant $C_2 > 0$ such that

$$\int_0^1 E^\alpha (-x + \chi_0) z_x z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' (x - \chi_0) z_x z_t dx dt \leq C_2 E(0)^\alpha E(t_1). \quad (3.7)$$

1 To end this part, we see that plugging (3.7) into (3.6) we arrive at

$$\begin{aligned}
& \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt \leq C_2 E(0)^\alpha E(t_1) \\
& + \int_{t_1}^{t_2} \int_0^1 E^\alpha (-x + \chi_0) z_x ag(z_t) dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt \\
& + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt. \quad (3.8)
\end{aligned}$$

2 **Part 2.** In this part we handle the right-hand side of (3.8). Let us take a non-negative function
3 $\chi_1 \in C_0^\infty(\omega)$ such that $\chi_1 = C_1$ in ω_0 . Note that this is possible because $\overline{\omega_0} \subset \omega$. Multiplying (1.2)
4 with (1.7) by $E(t)^\alpha \chi_1(x) z(t, x)$ and then performing some integrations by parts on $(t_1, t_2) \times (0, 1)$ we
5 obtain

$$\begin{aligned}
C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_{xx}|^2 + |z_{xxx}|^2) dx dt & \leq - \int_0^1 E^\alpha \chi_1 z z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_1 z z_t dx dt \\
& - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1'' z + 2\chi_1' z_x) z_{xx} dx dt - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1''' z + 3\chi_1'' z_x + 3\chi_1' z_{xx}) z_{xxx} dx dt \\
& - \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_1 z ag(z_t) dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_1 |z_t|^2 dx dt. \quad (3.9)
\end{aligned}$$

6 To handle the first four terms of the right-hand side of (3.9) we can proceed as we did for the
7 obtention of (3.7) to get a constant $C_3 > 0$ such that

$$\begin{aligned}
& - \int_0^1 E^\alpha \chi_1 z z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_1 z z_t dx dt - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1'' z + 2\chi_1' z_x) z_{xx} dx dt \\
& - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1''' z + 3\chi_1'' z_x + 3\chi_1' z_{xx}) z_{xxx} dx dt \leq C_3 E(0)^\alpha E(t_1) \\
& + \frac{1}{2} \int_{t_1}^{t_2} \int_0^1 E^\alpha |z_{xxx}|^2 dx dt + C_3 \int_{t_1}^{t_2} \int_\omega E^\alpha (|z|^2 + |z_x|^2 + |z_{xx}|^2) dx dt. \quad (3.10)
\end{aligned}$$

8 Accordingly, plugging (3.10) into (3.9), and then, the resulting expression into (3.8), we find a
9 constant $C_4 > 0$ such that

$$\begin{aligned}
& \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(|z_{xx}|^2 + \frac{3}{2} |z_{xxx}|^2 \right) dx dt \leq C_4 E(0)^\alpha E(t_1) \\
& + C_4 \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z| + |z_x|) |ag(z_t)| dx dt + C_4 \int_{t_1}^{t_2} \int_\omega E^\alpha |z_t|^2 dx dt \\
& + C_4 \int_{t_1}^{t_2} \int_\omega E^\alpha (|z|^2 + |z_x|^2 + |z_{xx}|^2) dx dt. \quad (3.11)
\end{aligned}$$

10 In order to handle the last term of the right-hand side of (3.11) we recall useful convexity
11 inequalities for Sobolev spaces that can be found in [4, Chapter IV, (7.37)]; namely, there exists a
12 constant $C_5 > 0$ such that

$$\int_{\omega} |z_x|^2 dx \leq \frac{1}{\varepsilon_1} \int_{\omega} |z_{xx}|^2 dx + C_5 \varepsilon_1 \int_{\omega} |z|^2 dx, \quad \varepsilon_1 > 0,$$

$$\int_{\omega} |z_{xx}|^2 dx \leq \frac{1}{\varepsilon_2} \int_{\omega} |z_{xxx}|^2 dx + C_5 \varepsilon_2^2 \int_{\omega} |z|^2 dx, \quad \varepsilon_2 > 0.$$

1 Considering $\varepsilon_1 = 2C_4 > 0$ and $\varepsilon_2 = C_4 > 0$, we apply the previous two convexity inequalities for
 2 Sobolev spaces in (3.11) and then take into account that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$ to get
 3 a constant $C_6 > 0$ such that

$$\begin{aligned} & \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^{\alpha} \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt \leq C_4 E(0)^{\alpha} E(t_1) \\ & + C_4 \int_{t_1}^{t_2} \int_0^1 E^{\alpha} (|z| + |z_x|) |ag(z_t)| dx dt \\ & + C_4 \int_{t_1}^{t_2} \int_0^1 E^{\alpha} a |z_t|^2 dx dt + C_6 \int_{t_1}^{t_2} \int_0^1 E^{\alpha} a |z|^2 dx dt. \end{aligned} \quad (3.12)$$

4 **Part 3.** In this part we handle the last term of the right-hand side of (3.12). To this end, we
 5 adapt an idea introduced by Conrad and Rao in [3] that consists in selecting a multiplier as the weak
 6 solution of a suitable elliptic problem. For each $t \in [t_1, t_2]$ let us consider

$$\begin{cases} \chi_{xxxx} - \chi_{xxxxx} = a(x)z(t, x), & x \in (0, 1), \\ \chi(t, 0) = \chi_x(t, 0) = \chi_{xx}(t, 0) = 0, \\ \chi(t, 1) = \chi_x(t, 1) = \chi_{xx}(t, 1) = 0. \end{cases} \quad (3.13)$$

7 On the one hand, since $a(\cdot)z(t, \cdot) \in L^2(0, 1)$ for each $t \in [t_1, t_2]$, the Lax–Milgram theorem tells us
 8 that (3.13) has a unique weak solution $\chi(t, \cdot) \in H_0^3(0, 1)$ satisfying

$$\int_0^1 |\chi_{xx}(t, x)|^2 dx + \int_0^1 |\chi_{xxx}(t, x)|^2 dx = \int_0^1 a(x)z(t, x)\chi(t, x) dx, \quad t \in [t_1, t_2].$$

9 Here we see that the Cauchy and Poincaré inequalities give us a constant $C_7 > 0$ such that

$$\int_0^1 |\chi(t, x)|^2 dx + \int_0^1 |\chi_x(t, x)|^2 dx \leq C_7 \int_0^1 |a(x)z(t, x)|^2 dx, \quad t \in [t_1, t_2]. \quad (3.14)$$

10 On the other hand, since $a(\cdot)z_t(t, \cdot) \in L^2(0, 1)$ for each $t \in [t_1, t_2]$, we can proceed as we just did for
 11 the obtention of (3.14) to conclude that $\chi_t(t, \cdot) \in H_0^3(0, L)$ satisfies

$$\int_0^1 |\chi_t(t, x)|^2 dx \leq C_7 \int_0^1 |a(x)z_t(t, x)|^2 dx, \quad t \in [t_1, t_2]. \quad (3.15)$$

12 Multiplying (1.2) with (1.7) by $E(t)^{\alpha}\chi(t, x)$, then performing some integrations by parts on
 13 $(t_1, t_2) \times (0, 1)$ and finally recalling that $\chi(t, x)$ is the unique weak solution of (3.13) we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_0^1 E^\alpha a|z|^2 dx dt &= - \int_0^1 E^\alpha \chi z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi z_t dx dt \\ &\quad - \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi a g(z_t) dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_t z_t dx dt. \end{aligned} \quad (3.16)$$

1 To handle the first two terms of the right-hand side of (3.16), we consider (3.14) and then proceed
 2 as we did for the obtention of (3.7) to get a constant $C_8 > 0$ such that

$$- \int_0^1 E^\alpha \chi z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi z_t dx dt \leq C_8 E(0)^\alpha E(t_1).$$

3 To end this part, we see that plugging the previous inequality into (3.16), and then, the resulting
 4 expression into (3.12), we find a constant $C_9 > 0$ such that

$$\begin{aligned} \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt &\leq C_9 E(0)^\alpha E(t_1) \\ + C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha (|\chi| + |z| + |z_x|) |a g(z_t)| dx dt & \\ + C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha a |z_t|^2 dx dt + C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha |\chi_t z_t| dx dt. & \end{aligned} \quad (3.17)$$

5 **Part 4.** In this part we handle the last three terms of the right-hand side of (3.17). In this part
 6 we consider a constant $C_{10} > 0$ that might vary from line to line. On the one hand, the continuous
 7 injection $H^1(0, 1) \hookrightarrow L^\infty(0, 1)$, (3.14) and Poincaré's inequality allow us to obtain

$$\int_{t_1}^{t_2} \int_0^1 E^\alpha (|z| + |z_x| + |\chi|) |a g(z_t)| dx dt \leq C_{10} \int_{t_1}^{t_2} \int_0^1 E^\alpha E^{\frac{1}{2}} |a g(z_t)| dx dt.$$

8 On the other hand, the Cauchy-Schwarz inequality and (3.15) imply

$$\int_{t_1}^{t_2} \int_0^1 E^\alpha a |z_t|^2 dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha |\chi_t z_t| dx dt \leq C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} \left(\int_0^1 |a z_t|^2 dx \right)^{\frac{1}{2}} dt.$$

9 Accordingly, combining (3.17) with the previous two inequalities lead us to

$$\begin{aligned} \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt &\leq C_{10} E(0)^\alpha E(t_1) \\ + C_{10} \int_{t_1}^{t_2} \int_0^1 E^\alpha E^{\frac{1}{2}} |a g(z_t)| dx dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} \left(\int_0^1 |a z_t|^2 dx \right)^{\frac{1}{2}} dt. & \end{aligned} \quad (3.18)$$

10 In order to handle the last two terms of the right-hand side of (3.18), for each $t \in [t_1, t_2]$ we
 11 introduce the sets $\Omega_1(t) = \{x \in (0, 1) / |z_t(t, x)| \leq 1\}$ and $\Omega_2(t) = \{x \in (0, 1) / |z_t(t, x)| > 1\}$. Setting
 12 $p_1 = R + 1 > 1$ and $p_2 = 1/S + 1 > 1$, we apply Hölder's inequality and then consider (A1), (A2),
 13 (1.8) and (3.3) to deduce

$$\begin{aligned}
\int_0^1 |ag(z_t)| dx &= \int_{\Omega_1(t)} |ag(z_t)| dx + \int_{\Omega_2(t)} |ag(z_t)| dx \\
&\leq C_{10} \left(\int_{\Omega_1(t)} |ag(z_t)| |z_t|^{\frac{p_1-1}{R}} dx \right)^{\frac{1}{p_1}} + C_{10} \left(\int_{\Omega_2(t)} |ag(z_t)| |z_t|^{S(p_2-1)} dx \right)^{\frac{1}{p_2}} \\
&\leq C_{10} (-E')^{\frac{1}{R+1}} + C_{10} (-E')^{\frac{S}{S+1}}, \quad t \in [t_1, t_2].
\end{aligned} \tag{3.19}$$

1 Similarly as we just did for the obtention of (3.19), we can also deduce

$$\begin{aligned}
\int_0^1 |az_t|^2 dx &= \int_{\Omega_1(t)} |az_t|^2 dx + \int_{\Omega_2(t)} |az_t|^2 dx \\
&\leq C_{10} \int_{\Omega_1(t)} |ag(z_t)z_t|^{\frac{2}{R+1}} dx + C_{10} \int_{\Omega_2(t)} |ag(z_t)z_t| dx \\
&\leq C_{10} (-E')^{\frac{2}{R+1}} + C_{10} (-E'), \quad t \in [t_1, t_2].
\end{aligned} \tag{3.20}$$

2 Accordingly, combining (3.18)–(3.20) yield

$$\begin{aligned}
\int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt &\leq C_{10} E(0)^\alpha E(t_1) \\
+ C_{10} \int_{t_1}^{t_2} E^{\alpha+\frac{1}{2}} (-E')^{\frac{1}{R+1}} dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{S}{S+1}} dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{1}{2}} dt. &\tag{3.21}
\end{aligned}$$

3 Let us set $p_3 = 1/R + 1 > 1$ with $q_3 = R + 1 > 1$ and $p_4 = S + 1 > 1$ with $q_4 = 1/S + 1 > 1$,
4 which satisfy that $1/p_3 + 1/q_3 = 1$ and $1/p_4 + 1/q_4 = 1$. Keeping in mind that (3.3) implies that
5 $E(t_2) \leq E(t_1) \leq E(0)$ whenever $0 \leq t_1 < t_2 < \infty$, we apply Young's inequality to obtain

$$\begin{aligned}
\int_{t_1}^{t_2} E^{\alpha+\frac{1}{2}} (-E')^{\frac{1}{R+1}} dt &\leq \frac{1}{p_3 \varepsilon_1} \int_{t_1}^{t_2} E^{p_3(\alpha+\frac{1}{2})} dt + \frac{1}{q_3 \varepsilon_1^{\frac{q_3}{p_3}}} \int_{t_1}^{t_2} (-E')^{\frac{q_3}{R+1}} dt \\
&\leq \frac{C_{10}}{\varepsilon_1} \int_{t_1}^{t_2} E^{(\frac{1}{R}+1)(\alpha+\frac{1}{2})} dt + C_{10} \varepsilon_1^R E(t_1), \quad \varepsilon_1 > 0,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
\int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{S}{S+1}} dt &\leq \frac{1}{p_4 \varepsilon_2} \int_{t_1}^{t_2} E^{\alpha+1} E^{\frac{p_4}{2}-1} dt + \frac{1}{q_4 \varepsilon_2^{\frac{q_4}{p_4}}} \int_{t_1}^{t_2} E^\alpha (-E')^{\frac{q_4 S}{S+1}} dt \\
&\leq \frac{C_{10}}{\varepsilon_2} E(0)^{\frac{S-1}{2}} \int_{t_1}^{t_2} E^{\alpha+1} dt + C_{10} \varepsilon_2^{\frac{1}{S}} E(0)^\alpha E(t_1), \quad \varepsilon_2 > 0,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
\int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{1}{2}} dt &\leq \frac{1}{2\varepsilon_1} \int_{t_1}^{t_2} E^{\alpha+1} dt + \frac{1}{2} \varepsilon_1 \int_{t_1}^{t_2} E^\alpha (-E') dt \\
&\leq \frac{1}{2\varepsilon_1} \int_{t_1}^{t_2} E^{\alpha+1} dt + \frac{1}{2} \varepsilon_1 E(0)^\alpha E(t_1), \quad \varepsilon_1 > 0.
\end{aligned} \tag{3.24}$$

1 Taking into account the combination of (3.21) with (3.22)–(3.24), we infer from (3.22) that we need
 2 to ask $\alpha \geq 0$ so that $(1/R + 1)(\alpha + 1/2) = \alpha + 1$, which only happens when $\alpha = (R - 1)/2 \geq 0$. Let
 3 us choose $\varepsilon_1 = \varepsilon > 0$ and $\varepsilon_2 = \varepsilon E(0)^{\frac{S-1}{2}} > 0$. Then, from (3.21)–(3.24) it follows

$$\int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + \int_{t_1}^{t_2} \int_0^1 E^{\frac{R-1}{2}} \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt$$

$$\leq \frac{C_{10}}{\varepsilon} \int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + C_{10} \left(E(0)^{\frac{R-1}{2}} + \varepsilon^R + \varepsilon^{\frac{1}{S}} E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + \varepsilon E(0)^{\frac{R-1}{2}} \right) E(t_1), \quad \varepsilon > 0.$$

4 Here the choice of $\varepsilon = 2C_{10} > 0$ gives us a constant $C_{11} > 0$ such that

$$\int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + \int_{t_1}^{t_2} \int_0^1 E^{\frac{R-1}{2}} (|z_{xx}|^2 + |z_{xxx}|^2) dx dt \leq C_{11} \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right) E(t_1).$$

5 Finally, the previous inequality allows us to obtain (1.10) and (1.11) by letting $t_2 \rightarrow \infty$ and then
 6 applying (3.2) with

$$\alpha = \frac{R-1}{2} \geq 0 \quad \text{and} \quad \beta = C_{11} \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right) > 0.$$

7 The proof of Theorem 1.1 is complete. ■

8 Acknowledgements

9 The author dedicates this work to his friends at “Fondation Marjolin”, in Paris, for all the great
 10 moments lived together during the academic year 2015–2016. This study has been partially supported
 11 by FONDECYT 1140741.

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