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# Stabilization of the heat equation with disturbance at the flux boundary condition

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In this paper, we address the problem of boundary stabilization of the heat equation subjected to an unknown disturbance, which is assumed to be acting at the flux boundary condition. By means of Lyapunov techniques and the use of the sign multivalued operator, which is responsible of rejecting the effects of the disturbance, we design a multivalued feedback law to obtain the exponential stability of the closed-loop system. The well-posedness of the closed-loop system, which is a differential inclusion, is shown with the maximal monotone operator theory.

**KEYWORDS**

boundary disturbance, exponential stability, feedback stabilization, heat equation, Lyapunov techniques

**MSC CLASSIFICATION**

35B40, 34G20, 35K05, 74K10, 93D15, 93D23

## 1 | INTRODUCTION

Let us consider the temperature $z = z(t, x)$ of a rod of mass density $\rho = \rho(x)$, specific heat $c = c(x)$, thermal conductivity $k = k(x)$ and length $L$. We assume that one of the extremities of the rod is insulated, whereas a known heat flux $u(t)$ is acting at the other extremity. At this latter extremity, we also assume that the heat flux is being perturbed by an unknown disturbance $d(t)$, which represents an unaccounted heat flux. Then, according to Haberman [1, chapter 1] or Çengel and Ghajar [2, chapter 2], the temperature can be modeled by

$$\begin{cases} \rho c z_t - (k z_x)_x = 0, & (t, x) \in (0, \infty) \times (0, L), \\ k(0) z_x(t, 0) = 0, & t \in (0, \infty), \\ -k(L) z_x(t, L) = u(t) + d(t), & t \in (0, \infty), \\ z(0, x) = z_0(x), & x \in (0, L). \end{cases} \tag{1.1}$$

Regarding the parameters of the model, we assume

**(P)** $\rho \in C([0, L])$ and there exists $(\rho_0, \rho_1) \in (0, \infty)^2$ such that $\rho_0 \leq \rho \leq \rho_1$ in $[0, L]$. $c \in C([0, L])$ and there exists $(c_0, c_1) \in (0, \infty)^2$ such that $c_0 \leq c \leq c_1$ in $[0, L]$. $k \in C^1([0, L])$ and there exists $(k_0, k_1) \in (0, \infty)^2$ such that $k_0 \leq k \leq k_1$ in $[0, L]$.

For a regular enough solution $z = z(t, x)$ of (1.1), we define its energy by

$$E(t) = \frac{1}{2} \int_0^L |z|^2 dx, \; t \in [0, \infty). \tag{1.2}$$

**TABLE 1** Stabilization of partial differential equations with disturbances.

| Equation | Distributed disturbance | Boundary disturbance |
|---|---|---|
| Heat | Guzmán and Prieur [6] | Kang and Guo [7] |
| Wave | Fu and Xu [8] | Previous works [9, 10] |
| Beam | Arias [11] | Previous works [12–14] |
| Schrödinger | – | Kang and Guo [15] |

The purpose of this paper is to exponentially stabilize (1.1) by means of a feedback law $u(t)$ that suppresses the effects of an unknown boundary disturbance $d(t)$. Being precise, we aim to show the existence of $(C, \omega) \in [1, \infty) \times (0, \infty)$ such that

$$E(t) \leq Ce^{-\omega t}E(0),\ t \in [0, \infty). \tag{1.3}$$

Regarding the undisturbed case, the problem under consideration can be solved by means of Lyapunov techniques, as those exposed in the literature [3–5] for instance. Indeed,

*Claim* 1. If the disturbance is zero, then the feedback law $u(t) = k_0 z(t, L)$ yield the existence of $(C, \omega) \in [1, \infty) \times (0, \infty)$ such that (1.3) is satisfied.

The arguments leading to the validity of Claim 1 are presented in Section 2, which actually follows from the analysis done there. Let us argue that in the undisturbed case such a feedback law does not longer work.

*Claim* 2. If the disturbance is not zero, then the feedback law $u(t) = k_0 z(t, L)$ does not longer work.

*Proof.* For the sake of the argument let us consider the case of constant parameters $\rho = c = k = 1$ in **(P)**. Thus, let us consider (1.1) with the feedback law $u(t) = z(t, L)$. Then, taking the disturbance $d(t) = -1$ and the initial condition $z_0(x) = 1$ it follows that the corresponding unique solution is $z(t, x) = 1$, which possesses positive energy as $t \to \infty$. Therefore, (1.3) cannot be satisfied. □

In view of Claim 2, we see that a new feedback law is required to solve the problem under consideration, one being able to reject the effects of disturbances, since the feedback law for the undisturbed case presented in Claim 1 is not robust with respect to disturbances.

The stabilization problem for partial differential equations subjected to unknown disturbances, acting either in the domain or at the boundary, has been object of recent interest. In Table 1, we present, without being exhaustive, some of the concerned literature.

In the works presented in Table 1, the effects of the disturbances were handled with the aid of the sign multivalued operator. In our case, we consider the sign multivalued operator sign : $\mathbb{R} \to 2^{\mathbb{R}}$ ($2^{\mathbb{R}}$ denotes the power set of $\mathbb{R}$) given by

$$\text{sign}(f) = \begin{cases} \frac{f}{|f|} & \text{if} \quad f \neq 0, \\ [-1, 1] & \text{if} \quad f = 0. \end{cases} \tag{1.4}$$

Regarding the disturbance, although it is assumed unknown, we ask it to satisfy the following two assumptions, which are the standard ones that can be found in the literature.

- **(A1)** There exists $D \in (0, \infty)$ such that $|d(t)| \leq D$ for every $t \in [0, \infty)$.
- **(A2)** $d \in W^{2,1}(0, \infty)$ and $d(0) = 0$.

On the one hand, assumption **(A1)** is required for the construction of a feedback law able to reject the effects of the disturbance. There $D \in (0, \infty)$ can be chosen as we see fit: the greater its value is, the more disturbances the feedback law will be able to reject. On the other hand, assumption **(A2)** is required to prove the well-posedness of the corresponding closed-loop system.

Our main result is the following one.

**Theorem 1.1.** *Let us assume* ***(P)***, ***(A1)*** *and* ***(A2)***. *Let us take an initial condition $z_0$ in $H^2(0, L)$ such that*

$$k(0)z_0'(0) = 0 \textit{ and } k(L)z_0'(L) + k_0 z_0(L) + D\textit{sign}(z_0(L)) \ni 0. \tag{1.5}$$

*Let us construct the multivalued feedback law*

$$u(t) = k_0 z(t, L) + D\textit{sign}(z(t, L)). \tag{1.6}$$

*Then, there exists a unique* $z = z(t,x)$ *in* $W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ *such that*

$$\begin{cases} \rho c z_t - (k z_x)_x = 0 \text{ for almost every } (t,x) \in (0,\infty)\times(0,L), \\ k(0)z_x(t,0) = 0 \text{ for every } t \in [0,\infty), \\ k(L)z_x(t,L) + k_0 z(t,L) + D\text{sign}(z(t,L)) + d(t) \ni 0 \text{ for every } t \in [0,\infty), \\ z(0,x) = z_0(x) \text{ for every } x \in [0,L]. \end{cases} \tag{1.7}$$

*Moreover, there exists* $(C,\omega)$ *in* $[1,\infty)\times(0,\infty)$ *such that*

$$\|z(t,\cdot)\|_{L^2(0,L)} \le Ce^{-\omega t}\|z_0\|_{L^2(0,L)} \text{ for every } t \in [0,\infty). \tag{1.8}$$

*Remark* 1.1. Let $L = 1$. Let us consider the case of constant parameter $k = 1$ in **(P)**. Let us consider $D = 1$ in **(A1)**. Then, there are plenty of initial conditions $z_0$ that fulfill (1.5). For instance,

(i) Let $z_0(x) = 2 - x^2$. Then, $k(0)z_0'(0) = 0$. Also, $k(L)z_0'(L) + k_0 z_0(L) + D\text{sign}(z_0(L)) \ni 0$ is fulfilled for $-2+1+1 = 0$ is true.

(ii) Let $z_0(x) = x^2(1-x)$. Then, $k(0)z_0'(0) = 0$. Also, $k(L)z_0'(L) + k_0 z_0(L) + D\text{sign}(z_0(L)) \ni 0$ is fulfilled for $[-2,0] \ni 0$ is true.

*Remark* 1.2. In (1.8), we have that $C = \left(\frac{\rho_1 c_1}{\rho_0 c_0}\right)^{1/2} \ge 1$ and $\omega = \frac{k_0}{M\rho_1 c_1} > 0$, where $M = 2L\max\{1, 2L\}$.

In Theorem 1.1 the multivalued feedback law (1.6) is constructed by means of Lyapunov techniques. The corresponding closed-loop system is the differential inclusion (1.7), and its well-posedness is shown with the maximal monotone operator theory, which may be consulted in previous works [16–18] for instance.

*Remark* 1.3. Let us note that Theorem 1.1 solves the exponential stabilization problem of the heat equation with variable coefficients subjected to the influence of a unknown boundary disturbance located at the flux boundary condition by means of a multivalued feedback law. The effects of the unknown boundary disturbance were handled with the aid of the sign multivalued operator.

The rest of this paper is organized as follows. The feedback design part of Theorem 1.1 is carried out in Section 2. Then, its well-posedness part is done in Section 3. Finally, numerical simulations are provided in Section 4.

## 2 | FEEDBACK DESIGN

Let us assume that $z = z(t,x)$ is a regular enough solution of (1.7). We proceed to perform formal computations, which later are justified in view of the analysis done in the next section. From (1.7) it follows, after one integration by parts, that

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L \rho c|z|^2 dx\right) + \int_0^L k|z_x|^2 dx - kz_x z|_{x=0}^{x=L} = 0. \tag{2.1}$$

In view of $k(0)z_x(t,0) = 0$ and $k(L)z_x(t,L) + k_0 z(t,L) + D\text{sign}(z(t,L)) + d(t) \ni 0$, we see that

$$-kz_x z|_{x=0}^{x=L} = -k(L)z_x(t,L)z(t,L) = k_0|z(t,L)|^2 + D\phi z(t,L) + d(t)z(t,L)\ \forall \phi \in \text{sign}(z(t,L)). \tag{2.2}$$

Let us recall that $\phi p = |p|$ for all $\phi \in \text{sign}(p)$ and **(A1)**. Plugging (2.2) into (2.1) and then taking into account **(P)**, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L \rho c|z|^2 dx\right) + k_0\left(|z(t,L)|^2 + \int_0^L |z_x|^2 dx\right) \le -D\phi z(t,L) - d(t)z(t,L) \le -D|z(t,L)| + |d(t)||z(t,L)| \le 0. \tag{2.3}$$

Considering **(P)** and that

$$\int_0^L |p|^2 dx \le M\left(|p(L)|^2 + \int_0^L |p_x|^2 dx\right)\ \forall p \in H^1(0,L),$$

where $M = 2L\max\{1, 2L\}$, it follows

$$k_0\left(|z(t,L)|^2 + \int_0^L |z_x|^2 dx\right) \geq \frac{k_0}{M}\int_0^L \frac{\rho c}{\rho c}|z|^2 dx \geq \frac{k_0}{M\rho_1 c_1}\int_0^L \rho c|z|^2 dx. \tag{2.4}$$

Finally, plugging (2.4) into (2.3), we get

$$\frac{d}{dt}\left(\int_0^L \rho c|z|^2 dx\right) + \frac{2k_0}{M\rho_1 c_1}\int_0^L \rho c|z|^2 dx \leq 0,$$

from which we arrive at (1.8) after considering **(P)**. Let us note that the constants $(C, \omega)$ in $[1,\infty) \times (0,\infty)$ of (1.8) are explicitly stated in Remark 1.2.

We have shown the feedback design part of Theorem 1.1.

## 3 | WELL-POSEDNESS

In this section, we prove the well-posedness part of Theorem 1.1. To that end, let us recall that the corresponding closed-loop system is given by (1.7). With the introduction of

$$y(t,x) = z(t,x) + \frac{1}{k(L)}d(t)\phi(x) \text{ with } \phi(x) = \frac{1}{2L}x^2 - \frac{L}{2}, \tag{3.1}$$

it follows from (1.7) that $y = y(t,x)$ satisfies

$$\begin{cases} y_t - \frac{1}{\rho c}(ky_x)_x = f, & (t,x) \in (0,\infty)\times(0,L), \\ k(0)y_x(t,0) = 0, & t \in (0,\infty), \\ k(L)y_x(t,L) + k_0 y(t,L) + D\text{sign}(y(t,L)) \ni 0, & t \in (0,\infty), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{3.2}$$

where

$$f(t,x) = \frac{1}{k(L)}d'(t)\phi(x) - \frac{1}{k(L)}d(t)\frac{1}{\rho(x)c(x)}(k(x)\phi'(x))' \text{ and } y_0(x) = z_0(x). \tag{3.3}$$

Let $\mathcal{H} = L^2(0,L)$ and let us denote by $(\cdot,\cdot)_{\mathcal{H}}$ its usual inner product. Let $(\cdot,\cdot)_w : \mathcal{H}\times\mathcal{H} \to [0,\infty)$ be the bilinear form given by $(u,v)_w = \int_0^L w(x)u(x)v(x)\,dx$, where $w \in C([0,L])$ is such that there exist $(w_0, w_1) \in (0,\infty)^2$ so that $w_0 \leq w(x) \leq w_1$ for all $x \in [0,L]$. Since $w_0\|p\|^2_{\mathcal{H}} \leq (p,p)_w \leq w_1\|p\|^2_{\mathcal{H}}$, it follows that $\mathcal{H}$ endowed with the inner product $(\cdot,\cdot)_w$ is also a Hilbert space, which is the one we consider in this section with the choice of $w = \rho c$.

Let us introduce the operator

$$\begin{cases} \mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}, \\ \mathcal{A}p = -\frac{1}{\rho c}(kp')', \\ D(\mathcal{A}) = \{p \in \mathcal{H} \,/\, \mathcal{A}p \in \mathcal{H},\, k(0)p'(0) = 0,\, k(L)p'(L) + k_0 p(L) + D\text{sign}(p(L)) \ni 0\}. \end{cases} \tag{3.4}$$

Let us note that $D(\mathcal{A})$ is not a linear subspace due to the sign multivalued operator, and hence, $\mathcal{A}$ is not a linear operator. Finally, we see that (3.2) can be written in operator form as follows:

$$\begin{cases} \frac{dy}{dt} + \mathcal{A}y = f,\ t \in [0,\infty), \\ y(0) = y_0. \end{cases} \tag{3.5}$$

In order to prove that (3.5) is well-posed, we apply the maximal monotone operator theory. In that direction, we have the following result.

**Proposition 3.1.** *The operator defined by (3.4) is a maximal monotone operator.*

*Proof.* In view of [17, chapter IV, lemma 1.3] (see also [18, theorem 2.2]), it suffices to prove that the operator $\mathcal{A}$ is monotone and that $R(I+\mathcal{A}) = \mathcal{H}$.

Let $(u, v) \in D(\mathcal{A}) \times D(\mathcal{A})$. Then, we have

$$\begin{aligned}(\mathcal{A}u - \mathcal{A}v, u - v)_w &= \int_0^L \rho c(\mathcal{A}u - \mathcal{A}v)(u - v)\,dx \\ &= \int_0^L k(u' - v')^2 dx - k(L)\left(u'(L) - v'(L)\right)(u(L) - v(L)).\end{aligned} \tag{3.6}$$

Since $(u, v) \in D(\mathcal{A}) \times D(\mathcal{A})$, there exist $\tilde{u} \in \operatorname{sign}(u(L))$ and $\tilde{v} \in \operatorname{sign}(v(L))$ such that $k(L)u'(L) + k_0u(L) + D\tilde{u} = 0$ and $k(L)v'(L) + k_0v(L) + D\tilde{v} = 0$, thus obtaining

$$-k(L)\left(u'(L) - v'(L)\right)(u(L) - v(L)) = k_0(u(L) - v(L))^2 + D(\tilde{u} - \tilde{v})(u(L) - v(L)). \tag{3.7}$$

Combining (3.6) and (3.7) it follows that $(\mathcal{A}u - \mathcal{A}v, u - v)_w \geq 0$ is a consequence of the monotonicity of the sign multivalued operator. Accordingly, the operator $\mathcal{A}$ is monotone.

Let us prove that $R(I+\mathcal{A}) = \mathcal{H}$. Given a $f \in \mathcal{H}$, we need to prove the existence of a $p \in D(\mathcal{A})$ such that $p + \mathcal{A}p = f$ for almost every $x \in (0, L)$, or equivalently $wp - (kp')' = wf$ for almost every $x \in (0, L)$. To that end, we proceed as in Conrad and Pierre [19] and analyze an optimization problem. Let us introduce the functional $J : H^1(0, L) \to \mathbb{R}$ by

$$J(p) = \frac{1}{2}\int_0^L \left(wp^2 + (kp')^2 - wfp\right)dx + \varphi_\lambda(p(L)), \tag{3.8}$$

where $\varphi_\lambda : \mathbb{R} \to \mathbb{R}$ is the Moreau regularization of the convex and continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = k_0x^2/2 + D|x|$. Thanks to [17, chapter IV, proposition 1.8] (see also [18, theorem 2.9]), we have

$$\varphi_\lambda(x) = \frac{\lambda}{2}|\alpha_\lambda(x)|^2 + \varphi(J_\lambda(x)),\ (\lambda, x) \in (0, \infty) \times \mathbb{R},$$

where $\alpha_\lambda : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of the maximal monotone operator $\alpha : \mathbb{R} \to 2^{\mathbb{R}}$ given by $\alpha(x) = (\partial\varphi)(x) = k_0x + D\operatorname{sign}(x)$ and $J_\lambda = (I + \lambda\alpha)^{-1} : \mathbb{R} \to \mathbb{R}$, which corresponds to the resolvent of $\alpha$. Furthermore, from the above-mentioned result, we also have that $\varphi_\lambda$ is a convex and differentiable function satisfying that $\varphi'_\lambda(x) = \alpha_\lambda(x)$.

The functional $J$ is convex and continuous. Let us note that the continuity holds thanks that the injection of $H^1(0, L)$ into $C([0, L])$ is continuous. Moreover, the functional $J$ is coercive. Indeed, considering the non-negativity of $\varphi_\lambda$ in (3.8), we get

$$J(p) \geq \frac{1}{2}\min\left\{\rho_0c_0, k_0^2\right\}\|p\|^2_{H^1(0,L)} - \rho_1c_1\|f\|_{L^2(0,L)}\|p\|_{H^1(0,L)},$$

from which we deduce the claim since $J(p) \to \infty$ as $\|p\|_{H^1(0,L)} \to \infty$. Therefore, theorem 2.19 in Peypouquet [20] allows us to infer the existence of a minimizer $m \in H^1(0, L)$ for the functional $J$. Due to proposition 3.20 in Peypouquet [20] and theorem 3.24 in Peypouquet [20], it follows that the Gâteaux derivative of the functional $J$ at such minimizer must vanish for each direction $d \in H^1(0, L)$. Accordingly, as $\varphi'_\lambda(x) = \alpha_\lambda(x)$, we obtain

$$J'(m; d) = \int_0^L \left(wmd + km'd' - wfd\right)dx + \alpha_\lambda(m(L))d(L) = 0\ \forall d \in H^1(0, L). \tag{3.9}$$

As far as (3.9) is concerned, the facts $C_0^\infty(0, L) \subset H^1(0, L)$ and $wm - wf \in L^2(0, L)$ imply that $(km')' = wm - wf$ in the sense of distributions, thus concluding that $km' \in H^1(0, L)$. Furthermore, after one integration by parts, we get

$$\int_0^L \left(wm - (km')' - wf\right)d\,dx = 0\ \forall d \in C_0^\infty(0, L),$$

which tells us that $wm - (km')' = wf$ for almost every $x \in (0, L)$. Then, after one integration by parts in (3.9), we arrive at

$$-k(0)m'(0)d(0) + [k(L)m'(L) + \alpha_\lambda(m(L))]d(L) = 0 \; \forall d \in H^1(0, L).$$

In summary,

**Lemma 3.1.** *For any $\lambda \in (0, \infty)$ there exists $p_\lambda \in H^1(0, L)$ such that $kp'_\lambda \in H^1(0, L)$, $wp_\lambda - (kp'_\lambda)' = wf$ for almost every $x \in (0, L)$, $k(0)p'_\lambda(0) = 0$ and $k(L)p'_\lambda(L) + \alpha_\lambda(p_\lambda(L)) = 0$.*

Let us consider the $p_\lambda$ given in Lemma 3.1. We proceed to prove that $R(I + \mathcal{A}) = \mathcal{H}$ by analyzing what happens to $p_\lambda$ as $\lambda \to 0^+$. In virtue of Lemma 3.1, we have

$$\int_0^L w|p_\lambda|^2 dx + \int_0^L k|p'_\lambda|^2 dx - k(L)p'_\lambda(L)p_\lambda(L) = \int_0^L \left(\sqrt{w} f\right)\left(\sqrt{w} p_\lambda\right) dx. \tag{3.10}$$

Since $0 \in \alpha(0)$, we get $J_\lambda(0) = 0$, and consequently, $\alpha_\lambda(0) = 0$. Then, as $\alpha_\lambda$ is also a maximal monotone operator, in particular from its monotonicity, we infer $-k(L)p'_\lambda(L)p_\lambda(L) = \alpha_\lambda(p_\lambda(L))p_\lambda(L) \geq 0$, which considered in (3.10) allows us to obtain, after applying Cauchy inequality,

$$\min\{\rho_0 c_0, k_0\} \|p_\lambda\|^2_{H^1(0,L)} \leq \rho_1 c_1 \|f\|^2_{L^2(0,L)}. \tag{3.11}$$

Similarly, by Lemma 3.1 and Cauchy inequality it follows

$$\int_0^L k|p''_\lambda|^2 dx = \int_0^L \left(wp_\lambda - k'p'_\lambda - wf\right) \frac{1}{\sqrt{k}} \left(\sqrt{k} p''_\lambda\right) dx \leq \int_0^L \frac{2}{k}\left(w^2|p_\lambda|^2 + (k')^2|p'_\lambda|^2 + w^2|f|^2\right) dx + \frac{1}{2}\int_0^L k|p''_\lambda|^2 dx,$$

from which we arrive at

$$k_0 \|p''_\lambda\|^2_{L^2(0,L)} \leq \frac{4}{k_0} \max\left\{\rho_1^2 c_1^2, \|k'\|^2_{L^\infty(0,L)}\right\}\left(\|p_\lambda\|^2_{H^1(0,L)} + \|f\|^2_{L^2(0,L)}\right). \tag{3.12}$$

The injection of $H^1(0, L)$ into $C([0, L])$ is continuous, and hence, there exists $C_I \in (0, \infty)$ such that $\|p\|_{C([0,L])} \leq C_I \|p\|_{H^1(0,L)}$ for any $p \in H^1(0, L)$. Then, by Lemma 3.1, we obtain

$$|\alpha_\lambda(p_\lambda(L))| = |k(L)p'_\lambda(L)| \leq k_1 \|p'_\lambda\|_{C([0,L])} \leq k_1 C_I \|p_\lambda\|_{H^2(0,L)}. \tag{3.13}$$

Therefore, in view of (3.11), (3.12), and (3.13), we see that the sequences $(p_\lambda)_{\lambda \geq 0} \subset H^2(0, L)$ and $(\alpha_\lambda(p_\lambda(L)))_{\lambda \geq 0} \subset \mathbb{R}$ are bounded, and hence, there exist $(p, g) \in H^2(0, L) \times \mathbb{R}$ and subsequences, which we denote by the same symbols, such that $p_\lambda \rightharpoonup p$ in $H^2(0, L)$ and $\alpha_\lambda(p_\lambda(L)) \to g$ in $\mathbb{R}$ as $\lambda \to 0^+$. Moreover, we infer that $p_\lambda \to p$ in $C^1([0, L])$ as $\lambda \to 0^+$ since the injection of $H^2(0, L)$ into $C^1([0, L])$ is compact, implying that $p'_\lambda(x) \to p'(x)$ in $\mathbb{R}$ as $\lambda \to 0^+$ for all $x \in [0, L]$.

Accordingly, Lemma 3.1 together with the previous analysis allows us to obtain:

**Lemma 3.2.** *There exists $p \in H^2(0, L)$ such that $wp - (kp')' = wf$ for almost every $x \in (0, L)$, $k(0)p'(0) = 0$ and $k(L)p'(L) + g = 0$.*

In view of Lemma 3.2, we see that in order to complete the proof of $R(I + \mathcal{A}) = \mathcal{H}$, we just need to prove that $g \in \alpha(p(L))$ and then recall that $\alpha(p(L)) = k_0 p(L) + D\text{sign}(p(L))$. To that end, we make use of several results that come from the fact that $\alpha$ is a maximal monotone operator. Let us recall that $p_\lambda(L) \to p(L)$ in $\mathbb{R}$ and $\alpha_\lambda(p_\lambda(L)) \to g$ in $\mathbb{R}$ as $\lambda \to 0^+$ (the subsequences). Then, since the Yosida approximation of $\alpha$ satisfies that $\alpha_\lambda(x) \in \alpha(J_\lambda(x))$ for all $x \in \mathbb{R}$ and $\alpha$ is a closed operator, it suffices to prove that $J_\lambda(p_\lambda(L)) \to p(L)$ in $\mathbb{R}$ as $\lambda \to 0^+$ to deduce that $g \in \alpha(p(L))$. Taking into account that $|J_\lambda(x_1) - J_\lambda(x_2)| \leq |x_1 - x_2|$ for all $(x_1, x_2) \in \mathbb{R}^2$ it follows

$$\begin{aligned} |J_\lambda(p_\lambda(L)) - p(L)| &= |J_\lambda(p_\lambda(L)) - J_\lambda(p(L)) + J_\lambda(p(L)) - p(L)| \\ &\leq |p_\lambda(L) - p(L)| + |J_\lambda(p(L)) - p(L)|. \end{aligned}$$

Finally, we deduce that $J_\lambda(p_\lambda(L)) \to p(L)$ in $\mathbb{R}$ as $\lambda \to 0^+$ because the resolvent of $\alpha$ satisfies that $J_\lambda(x) \to x$ in $\mathbb{R}$ as $\lambda \to 0^+$ for all $x \in \mathbb{R}$. The proof of Proposition 3.1 is complete. □

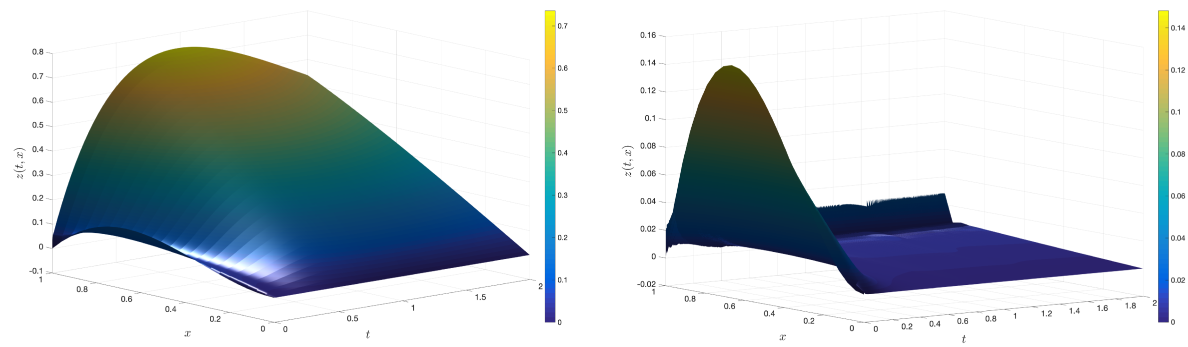
Let us finish the proof of Theorem 1.1.

*Proof of Theorem 1.1.* From (3.1) and (3.3), we have that $f \in W^{1,1}(0,\infty; C([0,L]))$ and $y_0 \in D(\mathcal{A})$. Then, chapter IV, theorem 4.1 in Showalter [17] gives the existence of a unique $y \in W^{1,1}(0,\infty; L^2(0,L))$ such that $y(0) = y_0$, $y'(t)+\mathcal{A}y(t) = f(t)$ for almost every $t > 0$ and $y(t) \in D(\mathcal{A})$ for every $t \geq 0$. Accordingly, thanks to (3.1) and (3.3), there exists a unique $z \in W^{1,1}(0,\infty; L^2(0,L)) \cap L^1(0,\infty; H^2(0,L))$ such that (1.7) is satisfied. Moreover, all the formal computations done in Section 2 make sense, implying that (1.8) is satisfied. The proof of Theorem 1.1 is complete. □

## 4 | NUMERICAL SIMULATIONS

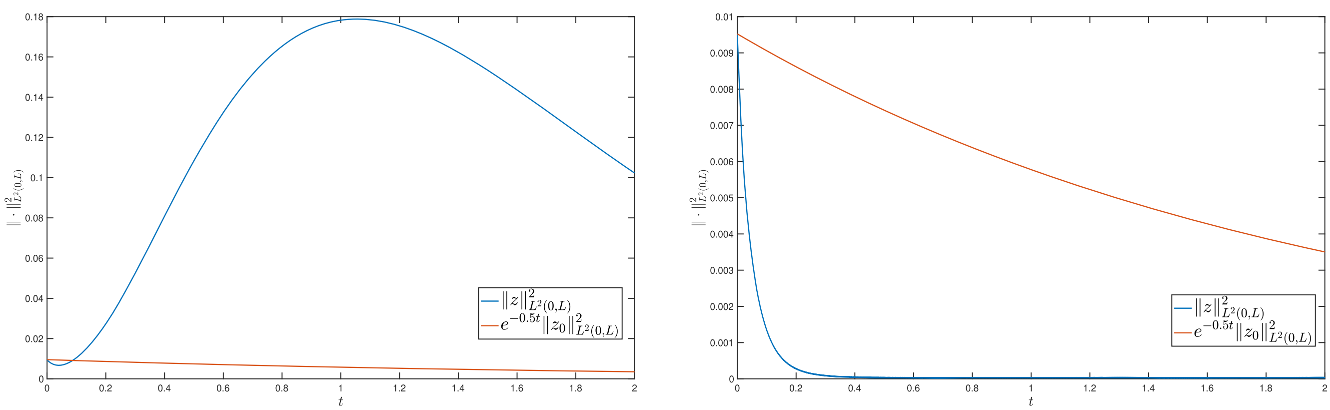
In this section, we present some numerical simulations in order to illustrate the stability result stated in Theorem 1.1. To that end, we discretize the system in space using a finite difference centered method and then solve the respective ODE system using the explicit Euler method. For simplicity, in this section, we set $L = 1$, and in **(P)**, we set the parameters of the model as $\rho = 1$, $c = 1$ and $k = 1$. We also consider $D = 1$ in **(A1)**.

In the first numerical simulation, we consider the initial condition $z_0(x) = x^2(1-x)$, $x \in [0,1]$, and the disturbance $d(t) = te^{-t}$, $t \in [0,\infty)$. Let us note that both the initial condition (see Remark 1.1) and the disturbance satisfy the hypotheses of Theorem 1.1. In Figure 1A, we show system (1.1) without control, that is with $u = 0$, while in Figure 1B, we show system (1.1) with control $u(t) = z(t,1) + \mathrm{sign}(z(t,1))$. We can observe from Figure 1A (uncontrolled) together with Figure 1B (controlled) that the control effectively rejects the influence of the disturbance, although the oscillation at



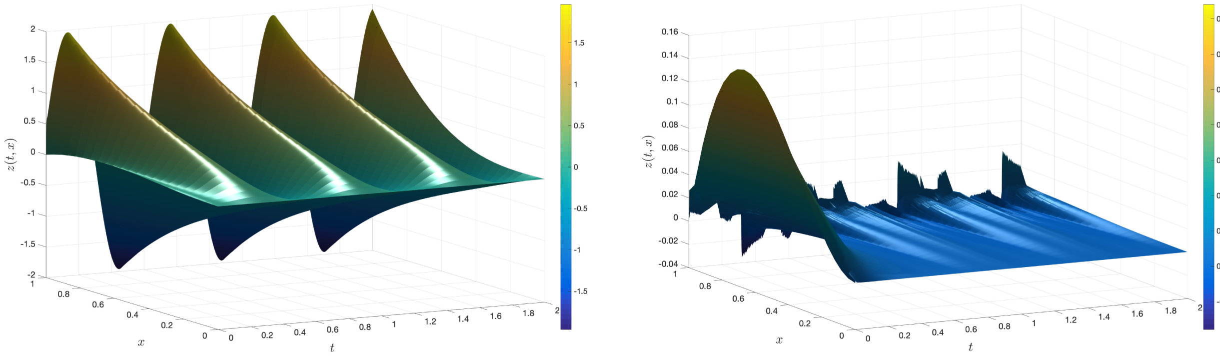

(A) Uncontrolled state. (B) Controlled state.

**FIGURE 1** The time evolution of the state $z(t,x)$ for the uncontrolled system (left) and the controlled system (right). [Colour figure can be viewed at wileyonlinelibrary.com]



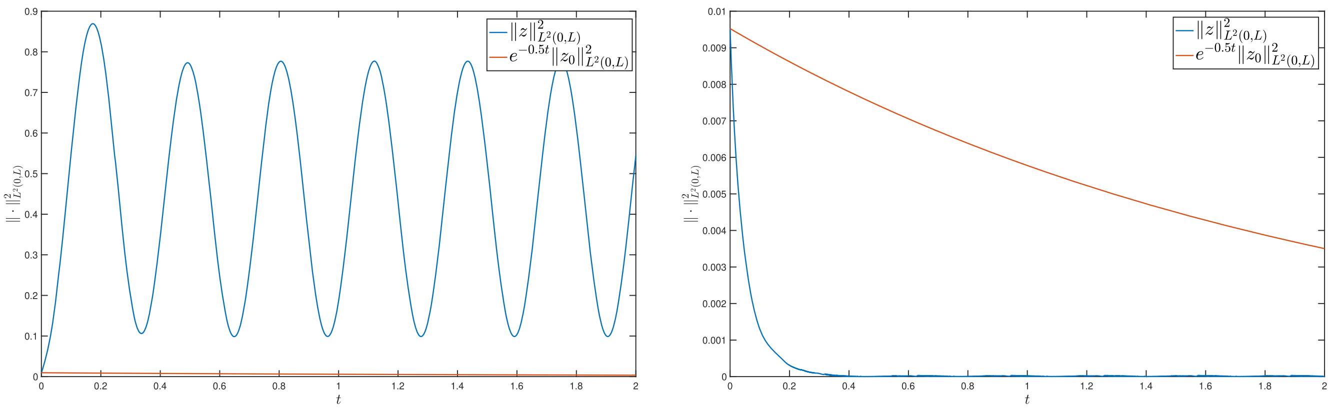

(A) Uncontrolled system. (B) Controlled system.

**FIGURE 2** For both the uncontrolled system (left) and the controlled system (right), we can find in blue the time evolution of $\|z(t,\cdot)\|^2_{L^2(0,L)}$ and in red the time evolution of $e^{-0.5t}\|z_0\|^2_{L^2(0,L)}$. [Colour figure can be viewed at wileyonlinelibrary.com]




(A) Uncontrolled state. (B) Controlled state.

**FIGURE 3** The time evolution of the state $z(t,x)$ for the uncontrolled system (left) and the controlled system (right). [Colour figure can be viewed at wileyonlinelibrary.com]




(A) Uncontrolled system. (B) Controlled system.

**FIGURE 4** For both the uncontrolled system (left) and the controlled system (right), we can find in blue the time evolution of $\|z(t,\cdot)\|^2_{L^2(0,L)}$ and in red the time evolution of $e^{-0.5t}\|z_0\|^2_{L^2(0,L)}$. [Colour figure can be viewed at wileyonlinelibrary.com]

$x = 1$ in Figure 1B (controlled) is expected in virtue of the $L^2$ norm in (1.8). Such oscillation would disappear if the norm in (1.8) were the supremum norm.

Due to the situation described in the first paragraph of this section, we have that $\omega = 1/4$ (see Remark 1.2). In Figure 2, we compare the quantities $\|z(t,\cdot)\|^2_{L^2(0,L)}$ and $e^{-2\omega t}\|z_0\|^2_{L^2(0,L)}$ for both the uncontrolled and controlled systems. For the uncontrolled system, we see in Figure 2A that (1.8) is not satisfied, while the expected behavior is met for the controlled system, as shown in Figure 2B.

Regarding the second and last numerical simulation, we still consider the same initial condition $z_0(x) = x^2(1-x)$, $x \in [0,1]$, but a different disturbance, which this time is $d(t) = \sin(10t)$, $t \in [0,\infty)$. This time the hypotheses of Theorem 1.1 are not meet because the disturbance does not satisfy **(A2)** since $d \notin W^{2,1}(0,\infty)$. In Figure 3A, we show system (1.1) without control while in Figure 3B, we show system (1.1) with control $u(t) = z(t,1) + \mathrm{sign}(z(t,1))$. As before, in this situation, we still have that $\omega = 1/4$ (see Remark 1.2). In Figure 4, we compare the quantities $\|z(t,\cdot)\|^2_{L^2(0,L)}$ and $e^{-2\omega t}\|z_0\|^2_{L^2(0,L)}$ for both the uncontrolled and controlled systems. From Figure 3 and Figure 4, we still observe the desired behavior, the one described in the previous numerical simulation.

In virtue of these two numerical simulations, we propose the following two open problems.

(i) Both Figure 2B and Figure 4B suggest that $\omega > 0$ can be improved, that $\omega > 0$ can be larger than the one presented in Remark 1.2. Thus, we think that an interesting open problem is: Can we obtain Theorem 1.1 for any $\omega > 0$? This would be a rapid stabilization result.

(ii) The second and last numerical simulation suggest that the result stated in Theorem 1.1 might also be valid if we consider a less regular disturbance. Thus, we think that another open problem is: Can we obtain Theorem 1.1 for less regular data (initial condition and disturbance)?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## CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.


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