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NULL CONTROLLABILITY OF THE STRUCTURALLY DAMPED WAVE EQUATION ON THE TWO-DIMENSIONAL TORUS*

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Abstract. We investigate the null controllability of the wave equation with a Kelvin–Voigt damping on the two-dimensional torus $\mathbb{T}^2$. We consider a distributed control supported in a moving domain $\omega(t)$ with a uniform motion at a constant velocity $c = (1, \zeta)$. The results we obtain depend strongly on the topological features of the geodesics of $\mathbb{T}^2$ with constant velocity c. When $\zeta \in \mathbb{Q}$, writing $\zeta = p/q$ with p, q relatively prime, we prove that the null controllability holds if roughly the diameter of $\omega(0)$ is larger than $1/p$ and if the control time is larger than q. We also prove that for almost every $\zeta \in \mathbb{R}_+ \setminus \mathbb{Q}$, and also for some particular values including, e.g., $\zeta = e$, the null controllability holds for any choice of $\omega(0)$ and for a sufficiently large control time. The proofs rely on a delicate construction of the weight function in a Carleman estimate which gets rid of a topological assumption on the control region often encountered in the literature. Diophantine approximations are also needed when ζ is irrational.

Key words. wave equation, Kelvin–Voigt damping, null controllability, parabolic-transport system, moving control, Carleman estimates, diophantine approximations

AMS subject classifications. 35Q74, 93B05, 93B07, 93C20

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1. Introduction.

We are concerned with the null controllability of a classical model of viscoelasticity, namely the wave equation with both viscous Kelvin–Voigt damping and frictional damping. We consider a distributed control supported in a moving domain $\omega(t)$. The system reads

$$y_{tt} - \Delta y - \nu \Delta y_t + b(x) y_t = 1_{\omega(t)}(x) u(x,t), \quad x \in \Omega, \ t \in (0,T), \tag{1.1}$$

$$y = 0, \quad x \in \partial\Omega, \ t \in (0,T), \tag{1.2}$$

$$y(x,0) = y_0(x), \quad y_t(x,0) = y_1(x), \quad x \in \Omega. \tag{1.3}$$

Here, Ω is a smooth, bounded, open set in $\mathbb{R}^N$ (where $N \geq 1$), $\nu > 0$ is a viscous constant, $b \in L^\infty(\Omega)$ is a function determining the frictional damping, and $u = u(x,t)$ stands for the control input. It is well known that system (1.1)–(1.3) fails to be null controllable when $\omega(t) \equiv \omega$ is a *fixed* open set in Ω with $\Omega \setminus \overline{\omega} \neq \emptyset$. This fact, noticed in [17] for $N = 1$ (for boundary controls) is due to the existence of a limit point in the spectrum of the adjoint system. The same obstruction occurs for the Benjamin–Bona–Mahony (BBM) equation (see [15, 19]):

$$y_t - y_{txx} + y_x + y y_x = 1_\omega(x) u(x,t). \tag{1.4}$$

To overcome this problem, Rosier and Zhang [19] suggested replacing the *fixed* control region ω in (1.4) by a *moving* control region $\omega(t)$ that is allowed to visit the whole

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domain. More precisely, they proved that the system

$$y_t - y_{txx} + y_x + yy_x = a(x-ct)u(x,t), \quad x \in \mathbb{T},\ t \in (0,T), \tag{1.5}$$

$$y(x,0) = y_0(x), \quad x \in \mathbb{T}, \tag{1.6}$$

where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the one-dimensional torus and $a \in C^\infty(\mathbb{T})$ is a nonnegative and not identically null function, is null controllable in time $T > 1/|c|$. Using the same kind of distributed control $a(x-ct)u(x,t)$, Martin, Rosier, and Rouchon [14] proved that the wave equation with structural damping

$$y_{tt} - y_{xx} - y_{txx} = a(x-ct)u(x,t), \qquad x \in \mathbb{T}, \tag{1.7}$$

is null controllable in time $T > 1/|c|$.

The case of an open set Ω in $\mathbb{R}^N$ was considered by Chaves-Silva, Rosier, and Zuazua [4]. The null controllability of (1.1)–(1.3) was derived when $\omega(t) = X(\omega_0, t, 0)$, where X is the flow generated by some vector field $f \in C([0,T], W^{2,\infty}(\mathbb{R}^N, \mathbb{R}^N))$; that is, X solves

$$\begin{cases} \frac{\partial X}{\partial t}(x,t,t_0) = f(X(x,t,t_0),t), \\ X(x,t_0,t_0) = x. \end{cases}$$

For instance, with the choice $f(x,t) = \dot\gamma(t)$ for some $\gamma \in C^1([0,T], \mathbb{R}^N)$, we obtain

$$X(x,t,t_0) = x + \gamma(t) - \gamma(t_0).$$

Actually, the results in [4] (and also those in [6]) were derived under the assumption that there exist a bounded, smooth, open set $\omega_0 \subset \mathbb{R}^N$, a curve $\Gamma \in C^\infty([0,T], \mathbb{R}^N)$, and two times t_1, t_2 with $0 \le t_1 \le t_2 \le T$ such that

(1.8) $\Gamma(t) \in X(\omega_0, t, 0) \cap \Omega \quad \forall t \in [0,T]$;

(1.9) $\overline{\Omega} \subset \cup_{t\in[0,T]} X(\omega_0,t,0) = \{X(x,t,0);\ x \in \omega_0,\ t \in [0,T]\}$;

(1.10) $\Omega \setminus \overline{X(\omega_0,t,0)}$ is nonempty and connected for $t \in [0,t_1] \cup [t_2,T]$;

(1.11) $\Omega \setminus \overline{X(\omega_0,t,0)}$ has two (nonempty) connected components for $t \in (t_1,t_2)$;

(1.12) $\forall \gamma \in C([0,T],\Omega),\ \exists t \in [0,T],\ \gamma(t) \in X(\omega_0,t,0)$.

More recently, Chaves-Silva, Zhang, and Zuazua [6] investigated the null controllability of a heat equation with memory by using a moving control, namely the system

$$y_t - \Delta y + \int_0^t M(t-s)y(s)ds = 1_{\omega(t)}(x)u(x,t), \quad x \in \Omega,\ t \in (0,T), \tag{1.13}$$

$$y = 0, \quad x \in \partial\Omega,\ t \in (0,T), \tag{1.14}$$

$$y(x,0) = 0, \quad x \in \Omega. \tag{1.15}$$

Assuming that (1.8)–(1.12) hold, that $M \in L^1(0,T)$, and picking an open set ω with $\overline{\omega_0} \subset \omega$, they proved that system (1.13)–(1.15) is null controllable in time T by taking $\omega(t) := X(\omega,t,0)$. See also [3, 5, 8, 13] for some other PDEs for which a moving control region is needed to get a controllability result.

It would be desirable to obtain the null controllability of (1.1)–(1.3) assuming only (1.9), but such a result (if true) is still not available in the literature. It is quite obvious that the conditions (1.10) and (1.11) are not consequences of (1.9). On the other hand, it was noticed in [4] that (1.8) and (1.12) are not implied by (1.9). The most conservative condition seems to be (1.11), for the natural situation when a small ball $\omega(t) = B(\gamma(t), \varepsilon)$ visits the whole domain Ω (with $\varepsilon \ll 1$ so that (1.11) fails) is not covered by the theory developed in [4, 6].

The aim of the paper is to obtain *almost sharp* results, as far as the control region is concerned, for the null controllability of the wave equation with both Kelvin–Voigt and frictional damping. We shall focus on the case of the simple domain $\Omega = (0,1)^N$ with periodic boundary conditions (i.e., we shall assume that $x \in \mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$), and we take for the flow X a uniform translation corresponding to $f(x,t) = c$ ($c \in \mathbb{R}^N$ being a constant vector), i.e.,

$$X(x,t,t_0) = x + (t - t_0)c.$$

Even if our results could be stated in any dimension $N \geq 1$, we shall restrict ourselves to the dimension $N = 2$, for the sake of simplicity. We may, without loss of generality, assume that the viscous constant is $\nu = 1$.

We are thus concerned with the null controllability of the system

$$y_{tt} - \Delta y - \Delta y_t + b(x)y_t = 1_{\omega(t)}(x)\, u(x,t), \quad x \in \mathbb{T}^2,\ t \in (0,T), \tag{1.16}$$

$$y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), \quad x \in \mathbb{T}^2. \tag{1.17}$$

We shall assume that

$$\omega(t) = \omega + tc, \quad t \in \mathbb{R}, \tag{1.18}$$

where $c = (c_1, c_2) \neq (0,0)$ is a constant velocity and $\omega \subset \mathbb{T}^2$ is any open set with $\overline{\omega_0} \subset \omega$, ω_0 being a nonempty open set. Without loss of generality, we can assume that

$$(0,0) \in \omega_0, \tag{1.19}$$

$$c = (1, \zeta), \tag{1.20}$$

where $\zeta \in \mathbb{R}_+$. We shall assume throughout that

$$\exists T > 0, \qquad \mathbb{T}^2 = \bigcup_{t \in [0,T]} \omega(t). \tag{1.21}$$

We shall denote by π the projection from $\mathbb{R}^2$ onto $\mathbb{T}^2$, defined as $\pi(x) = x + \mathbb{Z}^2$ for all $x \in \mathbb{R}^2$.

Our aim is the derivation of null controllability results for (1.16)–(1.17) with $\omega(t)$ as in (1.18), that are sharp as far as the geometry of ω_0 is concerned. We shall investigate three possible cases: (i) $c = (1,0)$; (ii) $c = (1,\zeta)$ with $\zeta \in \mathbb{Q}_+^*$; (iii) $c = (1,\zeta)$ with $\zeta \in \mathbb{R}_+ \setminus \mathbb{Q}$.

Let us state the main results of this paper.

(i) Assume that $c = (1,0)$. Note that the condition (1.21) yields

$$\{x_2 \in \mathbb{T};\ \exists x_1 \in \mathbb{T},\ (x_1,x_2) \in \omega_0\} = \mathbb{T}.$$

In other words, the projection along the x_2-axis of the control region at $t = 0$ should be the whole domain $\mathbb{T}$. We shall see that, for an open set ω_0 which is roughly delimited by two curves $x_1 = \rho_i(x_2)$, $i = 1,2$, this condition is also sufficient.

The first result in this paper is a null controllability result for (1.16)–(1.17) when $c = (1,0)$.

THEOREM 1.1. *Assume that $c = (1, 0)$. Assume given two functions $\rho_1, \rho_2 \in C([0, 1], \mathbb{R})$ such that $\rho_i(0) = \rho_i(1)$ for $i = 1, 2$, and $0 < \rho_1(x_2) < \rho_2(x_2) < 1$ for $x_2 \in [0, 1]$, and let*

$$\omega_0 = \pi\{x = (x_1, x_2) \in [0, 1]^2;\ \rho_1(x_2) < x_1 < \rho_2(x_2)\}.$$

Pick any time $T > 1 - \min_{x_2\in[0,1]} \rho_2(x_2) + \max_{x_2\in[0,1]} \rho_1(x_2)$ Let ω be any open set in $\mathbb{T}^2$ such that $\overline{\omega_0} \subset \omega$, and let $(\omega(t))_{t\in[0,T]}$ be as in (1.18). Then for any $(y_0, y_1) \in [L^2(\mathbb{T}^2)]^2$ with $y_1 - \Delta y_0 \in L^2(\mathbb{T}^2)$, there exists a control $u \in L^2(0, T, L^2(\mathbb{T}^2))$ such that the solution y of (1.16)–(1.17) satisfies $y(., T) = y_t(., T) = 0$.

Note that the control time is sharp when the functions ρ_1 and ρ_2 assume constant values, for we get the condition $T > 1 - (\rho_2 - \rho_1)$.

(ii) Assume now that $c = (1, \frac{p}{q})$ with

$$p, q \in \mathbb{N}^*, \quad p \text{ and } q \text{ being relatively prime.} \tag{1.22}$$

Then it is well known that the curve $t \to \pi(ct)$ is q-periodic (hence closed), and that its image $\pi\{ct;\ t \in [0, q]\}$ is homeomorphic to a 1-sphere (i.e., a circle). It is compact and not dense in $\mathbb{T}^2$.

It is expected that the extension of ω_0 along the transversal variable x_1 should be sufficiently large for (1.16)–(1.17) to be null controllable. A precise statement of this claim is given in the following proposition.

PROPOSITION 1.1. *Assume that $c = (1, \frac{p}{q})$ with p and q as in (1.22). Let L and ε be positive numbers such that the set*

$$\omega_0 = \pi\big([0, L] \times \{0\} + B((0, 0), \varepsilon)\big)$$

satisfies

$$\bigcup_{t\in[0,T]} (\omega_0 + tc) = \mathbb{T}^2 \tag{1.23}$$

for some number $T > 0$. Then

$$L \ge \frac{1 - 2\sqrt{p^2 + q^2}\,\varepsilon - \pi\varepsilon^2}{p + 2\varepsilon}. \tag{1.24}$$

In particular, $L \ge \frac{1}{p} + O(\varepsilon)$ as $\varepsilon \to 0^+$.

Thus, the extension of ω_0 along the x_1-variable should be at least of order $1/p$. Conversely, we shall prove that under a similar condition the null controllability of (1.16)–(1.17) in large time can be derived. This is the content of the following theorem, which is the first main result of this paper.

THEOREM 1.2. *Assume that $c = (1, \frac{p}{q})$ where p and q are as in (1.22). Assume that $\omega_0 \subset \mathbb{T}^2$ is an open set with*

$$\pi\Big(\Big[0, \frac{1}{p}\Big] \times \{0\}\Big) \subset \omega_0.$$

Let ω be any open set in $\mathbb{T}^2$ such that $\overline{\omega_0} \subset \omega$, and let $(\omega(t))_{t\in[0,T]}$ be as in (1.18). Pick any $T > q$. Then for any $(y_0, y_1) \in [L^2(\mathbb{T}^2)]^2$ with $y_1 - \Delta y_0 \in L^2(\mathbb{T}^2)$, there exists a control $u \in L^2(0, T, L^2(\mathbb{T}^2))$ such that the solution y of (1.16)–(1.17) satisfies $y(., T) = y_t(., T) = 0$.

Remark 1.1.
1. Theorem 1.2 is sharp as far as both the *control domain* and the *time control* are concerned. For the control domain, this follows from Proposition 1.1, and for the time control, this follows from the observation that

$$\pi\Big\{s\Big(\frac{1}{p},1\Big)+t\Big(1,\frac{p}{q}\Big);\ s\in[0,1],\ t\in[0,T]\Big\}=\mathbb{T}^2 \iff T\geq q.$$

2. Theorem 1.2 applies when $\omega_0=B(0,r)$ for $r>(2p)^{-1}$ (the position of the center of the ball being irrelevant).

(iii) Assume that $c=(1,\zeta)$ with $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$. Then it is well known that the curve $t\to\pi(tc)$ is one-to-one on $\mathbb{R}$, and that its image $\{\pi(tc);\ t\in\mathbb{R}\}$ is *dense* in $\mathbb{T}^2$. It is thus natural to expect that for *any* nonempty open set $\omega_0\subset\mathbb{T}^2$, the set

$$\omega_0+[0,T]c=\{x+tc;\ x\in\omega_0,\ t\in[0,T]\}$$

covers $\mathbb{T}^2$ if the control time $T>0$ is large enough.

The following proposition shows that it is indeed the case.

Proposition 1.2. *If $c=(1,\zeta)$ with $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$, then for any nonempty open set $\omega_0\subset\mathbb{T}^2$, there exists some $T>0$ such that*

$$\omega_0+[0,T]c=\mathbb{T}^2. \tag{1.25}$$

The above proposition suggests that the control system (1.16)–(1.17) may be null controllable for *any* control region but in *large* time. Such a result can be established for almost every $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$. To state it, we need to introduce a class of irrational numbers that can be approximated by rational numbers with an error less than δ/q^2.

Definition 1.1. *For any $\delta>0$, let $\mathcal{I}_\delta$ denote the set of numbers $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$ such that for all $A>0$, there exists a pair (p,q) as in* (1.22) *with $q\geq A$ and*

$$\left|\zeta-\frac{p}{q}\right|\leq\frac{\delta}{q^2}. \tag{1.26}$$

It is well known that $\mathcal{I}_{1/\sqrt5}=\mathbb{R}_+\setminus\mathbb{Q}$, while $(\sqrt5-1)/2\notin\mathcal{I}_\delta$ for $\delta<1/\sqrt5$ (see, e.g., [11, Theorems 193 and 194]). Furthermore, it is known that $(\mathbb{R}_+\setminus\mathbb{Q})\setminus\mathcal{I}_\delta$ is of measure zero for any $\delta>0$ (see, e.g., [11, Theorem 196]).

The following result is the second main result in this paper.

Theorem 1.3. *Assume that $c=(1,\zeta)$ where $\zeta\in\mathcal{I}_\delta$ for some $\delta>0$ with*

$$f_2(\delta,\zeta):=\left(32+\frac{8}{\zeta}\right)\delta+\left(549+\frac{64}{\zeta}+\frac{8}{\zeta^2}\right)\delta^2<1.$$

Let ω be any nonempty open set in $\mathbb{T}^2$, and let $(\omega(t))_{t\in[0,T]}$ be as in (1.18). *Then there exists a time $T>0$ such that for any $(y_0,y_1)\in[L^2(\mathbb{T}^2)]^2$ with $y_1-\Delta y_0\in L^2(\mathbb{T}^2)$, there exists a control $u\in L^2(0,T,L^2(\mathbb{T}^2))$ such that the solution y of* (1.16)–(1.17) *satisfies $y(.,T)=y_t(.,T)=0$.*

Remark 1.2.
1. Theorem 1.3 applies for a.e. $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$, for the set $(\mathbb{R}_+\setminus\mathbb{Q})\setminus\cap_{n\geq1}\mathcal{I}_{1/n}$ is of measure zero.
2. Theorem 1.3 also applies when $\zeta\in\mathcal{I}_\delta$ with $\zeta\geq1$ and $\delta\leq10^{-2}$. Indeed, $f_2(\delta,\zeta)\leq f_2(\delta,1)=40\delta+621\delta^2$. The issue whether Theorem 1.3 is actually true for *any* $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$ is open and challenging.

3. We can provide *explicit examples* of numbers to which Theorem 1.3 applies. For a given $\zeta \in \mathbb{R}_+ \setminus \mathbb{Q}$, we introduce its *continued fraction* (see, e.g., [11]) $[a_0, a_1, a_2, \ldots]$ where $a_0 \in \mathbb{N}$, $a_i \in \mathbb{N}^*$ for $i \geq 1$ and $\zeta = \lim_{n\to+\infty}[a_0, a_1, \ldots, a_n]$ with
$$[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}\cdot$$
Denoting $[a_0, a_1, \ldots, a_n] = p_n/q_n$ with p_n and q_n as in (1.22), then $q_n \to \infty$ and
$$\left|\zeta - \frac{p_n}{q_n}\right| \leq \frac{1}{a_{n+1} q_n^2}\cdot$$
(See, e.g., [11, section 10.9].) Thus, $\xi \in \mathfrak{I}_\delta$ if $a_n > \delta^{-1}$ for infinitely many n. Let us give some examples of numbers ζ to which Theorem 1.3 applies. The first one is an algebraic number, the second and third ones are transcendental numbers.
   (a) Let $\zeta = \sqrt{m^2+1}$ with $m \in \mathbb{N}$. Then $\zeta = [m, 2m, 2m, 2m, \ldots]$ so that $\zeta \in \mathfrak{I}_\delta$ if $2m > \delta^{-1}$, and hence Theorem 1.3 applies for $m > 50$.
   (b) Let $\zeta = L = \sum_{k=1}^{\infty} 10^{-k!}$ (Liouville's constant). Then $L \in \mathfrak{I}_\delta$ for all $\delta > 0$. Indeed, setting $q_n = 10^{n!}$ and $p_n = \sum_{k=1}^{n} 10^{n!-k!}$ for $n \in \mathbb{N}^*$, for given $A, \delta > 0$ we notice that (p_n, q_n) is (for n large enough) as in (1.22) with $q_n > A$ and
$$\left|L - \frac{p_n}{q_n}\right| = \sum_{k=n+1}^{\infty} 10^{-k!} \leq \frac{2}{10^{(n+1)!}} \leq \frac{\delta}{q_n^2}\cdot$$
   (c) Let $\zeta = e = \exp(1)$. Then Euler proved that $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$ (see, e.g., [12]). Thus, $a_{3n-1} = 2n$ for all $n \in \mathbb{N}^*$, so that $e \in \mathfrak{I}_\delta$ for all $\delta > 0$.
4. Letting $\Omega(t) := \omega(t) \times \mathbb{T}^{N-2}$ and substituting $1_{\Omega(t)}$ to $1_{\omega(t)}$ in (1.16), we can easily derive null controllability results for the structurally damped wave equation with a moving control in $\mathbb{T}^N$. However, more interesting is the situation when the control region is of the form (1.18) with ω a small set (e.g., a small ball). It is likely that results similar to Theorems 1.1, 1.2, and 1.3 could be obtained in any dimension $N \geq 3$, although the extension of Proposition 2.3 (see below) to the dimension $N \geq 3$ seems to be hard. We noticed that the results in this paper depended strongly on the fact that the trajectory $\{\pi(tc);\ t \in \mathbb{R}\}$ (with $c = (1, \zeta)$) was periodic or dense in $\mathbb{T}^2$. Assume, e.g., that $N = 3$, and set $c = (1, \zeta_2, \zeta_3)$, where $\zeta_2, \zeta_3 \in \mathbb{R}_+$. Then it can be seen that the trajectory $\{\pi(tc);\ t \in \mathbb{R}\}$ is periodic (resp., is dense in $\mathbb{T}^3$) if and only if ζ_2 and ζ_3 are rational (resp., ζ_2, ζ_3, and ζ_2/ζ_3 are irrational). See [2, p. 92].

Let us say a few words about the proofs of Theorems 1.1, 1.2, and 1.3. Following [4], we decompose (1.16)–(1.17) into a system coupling a parabolic equation with an ordinary differential equation (ODE), namely

$$y_t - \Delta y + (b(x) - 1)y = z, (t, x) \in (0, T) \times \mathbb{T}^2, \tag{1.27}$$
$$z_t + z = 1_{\omega(t)}(x)u + (b(x) - 1)y, (t, x) \in (0, T) \times \mathbb{T}^2, \tag{1.28}$$
$$y(x, 0) = y_0(x), \quad x \in \mathbb{T}^2, \tag{1.29}$$
$$z(x, 0) = y_1(x) - \Delta y_0(x) + (b(x) - 1)y_0(x), \quad x \in \mathbb{T}^2. \tag{1.30}$$

Introducing the new unknown functions

$$h(x,t) := y(x+tc,t),$$
$$k(x,t) := z(x+tc,t),$$

we see that system (1.27)–(1.30) is transformed into the system

$$h_t - \Delta h - c\cdot\nabla h + (b(x+tc)-1)h = k, (t,x)\in(0,T)\times\mathbb{T}^2, \tag{1.31}$$
$$k_t - c\cdot\nabla k + k = 1_{\omega_0}(x)u(x+tc,t) + (b(x+tc)-1)h, (t,x)\in(0,T)\times\mathbb{T}^2, \tag{1.32}$$
$$h(x,0) = y_0(x), \quad x\in\mathbb{T}^2, \tag{1.33}$$
$$k(x,0) = y_1(x) - \Delta y_0(x) + (b(x)-1)y_0(x), \quad x\in\mathbb{T}^2 \tag{1.34}$$

for which the control input is supported in the *fixed domain* ω_0.

Next, the corresponding adjoint system to (1.31)–(1.34) reads

$$-v_t - \Delta v + c\cdot\nabla v + (b(x+tc)-1)v = (b(x+tc)-1)w, (t,x)\in(0,T)\times\mathbb{T}^2, \tag{1.35}$$
$$-w_t + c\cdot\nabla w + w = v, (t,x)\in(0,T)\times\mathbb{T}^2, \tag{1.36}$$
$$v(x,T) = v_T(x), \quad x\in\mathbb{T}^2, \tag{1.37}$$
$$w(x,T) = w_T(x), \quad x\in\mathbb{T}^2. \tag{1.38}$$

By classical duality arguments, the null controllability of (1.31)–(1.34) is proved whenever we have established the following observability inequality for the adjoint system (1.35)–(1.38):

$$\int_{\mathbb{T}^2}\big(|v(x,0)|^2 + |w(x,0)|^2\big)dx \le C_{obs}\int_0^T\int_{\omega_0}|w(x,t)|^2dxdt \tag{1.39}$$

for some constant $C_{obs} > 0$ and all $v_T, w_T \in L^2(\mathbb{T}^2)$. In order to establish the observability inequality (1.39), following [1, 4], we derive Carleman estimates for the backward heat equation (1.35) and for the transport equation (1.36) with the *same* weights functions. To get "almost sharp" results for the geometry of the control region, the construction of the weight functions in the Carleman estimates turns out to be much more delicate than in [4]. To derive Theorem 1.2, we need to prove the existence of a function $\psi_0\in C^\infty(\mathbb{T}^2)$ such that

$$\nabla\psi_0(x)\neq 0 \quad \forall x\in\mathbb{T}^2\setminus\omega_0, \tag{1.40}$$
$$c\cdot\nabla(c\cdot\nabla\psi_0)(x) > 0 \quad \forall x\in\mathbb{T}^2\setminus\omega_0. \tag{1.41}$$

This is done by viewing ψ_0 as a smooth $\mathbb{Z}^2$-periodic function which is constructed explicitly on the domain

$$D := \Big\{s\Big(\frac{1}{p},0\Big) + t\Big(1,\frac{p}{q}\Big),\ s\in[0,1),\ t\in[0,q)\Big\},$$

and by noticing that D is a *fundamental domain* for the action of the group $(\mathbb{Z}^2,+)$; that is to say, the family $\big(D+(m,n)\big)_{(m,n)\in\mathbb{Z}^2}$ constitutes a *partition* of $\mathbb{R}^2$.

The proof of Theorem 1.3 cannot be done along the same lines as for Theorem 1.2, for no fundamental domain similar to D is available. However, approximating $\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$ by some rational number p/q as in (1.26) and using Theorem 1.2, we can construct a function ψ_0 satisfying (1.40)–(1.41).

A few words should be said about what we mean by a solution of system (1.16)–(1.17) or, equivalently, by a solution of system (1.27)–(1.30). The following result follows from the classical semigroup theory (see, e.g., [16]). Its proof is omitted.

Proposition 1.3. *The operator*

$$A(y,z) = (\Delta y - (b(x)-1)y + z, -z + (b(x)-1)y),$$

with domain $D(A) = H^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2) \subset [L^2(\mathbb{T}^2)]^2$ *generates a strongly semigroup* $(e^{tA})_{t\geq 0}$ *in* $[L^2(\mathbb{T}^2)]^2$. *As a consequence, for any* $(y_0,z_0)\in[L^2(\mathbb{T}^2)]^2$ *and any* $u\in L^2(0,T,L^2(\mathbb{T}^2))$, *setting* $F(x,t)=(0,1_{\omega(t)}(x)\,u(x,t))$, *we have that the system*

$$(y,z)_t = A(y,z)+F, \quad (y,z)(.,0) = (y_0,z_0),$$

admits a unique mild solution $(y,z)\in C([0,T],[L^2(\mathbb{T}^2)]^2)$. *Furthermore,* $y\in L^2(0,T;H^1(\mathbb{T}^2))$.

Going back to system (1.16)–(1.17) and using (1.27), we see that for any $(y_0,y_1)\in H^2(\mathbb{T}^2)\times L^2(\mathbb{T}^2)$ and any control input $u\in L^2(0,T,L^2(\mathbb{T}^2))$, we can construct a solution $y\in C([0,T],L^2(\mathbb{T}^2))$ of (1.16)–(1.17) with $y_t-\Delta y\in C([0,T],L^2(\mathbb{T}^2))$.

This paper is organized as follows. Section 2 is devoted to the construction of a function ψ_0 as in (1.40)–(1.41) in each of the cases (i), (ii), and (iii). In section 3, we state and prove some Carleman estimates for a transport equation and for a backward heat equation with the same weights involving the function ψ_0 in their expressions. In section 4, we derive the observability inequality for the adjoint system (1.35)–(1.38) and next complete the proofs of Theorems 1.1, 1.2, and 1.3.

2. Construction of the function ψ_0. In this section, we prove the existence of a function $\psi_0\in C^\infty(\mathbb{T}^2)$ satisfying (1.40)–(1.41) for c and ω_0 as in Theorems 1.1, 1.2, and 1.3, respectively. Let us consider successively the cases (i) $c=(1,0)$; (ii) $c=(1,p/q)$ ($p,q\in\mathbb{N}^*$); and (iii) $c=(1,\zeta)$ ($\zeta\in\mathbb{R}_+\setminus\mathbb{Q}$).

(i) $c=(1,0)$.

Proposition 2.1. *Let* c *and* ω_0 *be as in Theorem* 1.1, *and let* $\tau > 1-\min_{x_2\in[0,1]}\rho_2(x_2) > 0$. *Then there exists a nonnegative function* $\psi_0\in C^\infty(\mathbb{T}^2)$ *satisfying*

$$2\Big(\tau - 1 + \min_{x_2\in[0,1]}\rho_2(x_2)\Big) \leq c\cdot\nabla\psi_0 \leq 2\Big(\tau + \max_{x_2\in[0,1]}\rho_1(x_2)\Big) \ \textit{in} \ \ \mathbb{T}^2\setminus\omega_0, \tag{2.1}$$

$$c\cdot\nabla(c\cdot\nabla\psi_0) = 2 \ \textit{in} \ \ \mathbb{T}^2\setminus\omega_0. \tag{2.2}$$

Proof. Let ρ_1 and ρ_2 be as in Theorem 1.1. Recall that $\rho_1(0)=\rho_1(1)$ and $\rho_2(0)=\rho_2(1)$. Introduce the open sets in $\mathbb{T}^2$:

$$\Omega_1 := \Big\{\pi(x);\ x=(x_1,x_2)\in[0,1]^2,\ \frac{2}{3}\rho_1(x_2)+\frac{1}{3}\rho_2(x_2) < x_1 < \frac{1}{3}\rho_1(x_2)+\frac{2}{3}\rho_2(x_2)\Big\},$$

$$\Omega_2 := \Big\{\pi(x);\ x=(x_1,x_2)\in[0,1]^2,\ \frac{4}{5}\rho_1(x_2)+\frac{1}{5}\rho_2(x_2) < x_1 < \frac{1}{5}\rho_1(x_2)+\frac{4}{5}\rho_2(x_2)\Big\}.$$

Since $\overline{\Omega_1}\subset\Omega_2$, we may pick (see, e.g., [9, Corollary 4.7]) a function $\chi\in C^\infty(\mathbb{T}^2)$ such that $0\leq\chi\leq 1$ and

$$\chi(x) = \begin{cases} 0 & \text{if } x\in\Omega_1,\\ 1 & \text{if } x\notin\Omega_2.\end{cases}$$

For $x = (x_1, x_2) \in [0,1]^2$, we set

$$\psi_0(x) = \begin{cases} \chi(\pi(x))\,(x_1+\tau)^2 & \text{if } x_1 \le \frac{1}{2}(\rho_1(x_2)+\rho_2(x_2)), \\ \chi(\pi(x))\,(x_1-1+\tau)^2 & \text{if } x_1 > \frac{1}{2}(\rho_1(x_2)+\rho_2(x_2)). \end{cases}$$

Then it is clear that ψ_0 can be extended as a $\mathbb{Z}^2$-periodic smooth function (i.e., $\psi_0 \in C^\infty(\mathbb{T}^2)$). On the other hand,

$$c\cdot\nabla\psi_0(x) = \frac{\partial\psi_0}{\partial x_1} = \begin{cases} 2(x_1+\tau) & \text{if } \ 0\le x_1 \le \rho_1(x_2), \\ 2(x_1-1+\tau) & \text{if } \ \rho_2(x_2) \le x_1 \le 1, \end{cases}$$

and hence

$$2\tau \le c\cdot\nabla\psi_0(x) \le 2(\tau + \max_{x_2\in[0,1]} \rho_1(x_2)) \ \text{ if } \ 0\le x_1\le\rho_1(x_2),$$
$$2\Big(\tau - 1 + \min_{x_2\in[0,1]}\rho_2(x_2)\Big) \le c\cdot\nabla\psi_0(x) \le 2\tau \ \text{ if } \ \rho_2(x_2)\le x_1\le 1,$$

so that (2.1) holds. Finally,

$$c\cdot\nabla(c\cdot\nabla\psi_0) = \frac{\partial^2\psi_0}{\partial x_1^2} = 2 \quad \text{if} \quad x_1 \notin (\rho_1(x_2),\rho_2(x_2)),$$

so that (2.2) holds as well. □

(ii) $c = (1, \frac{p}{q})$, $p, q \in \mathbb{N}^*$, p and q relatively prime.

Before performing the construction of the function ψ_0, we first prove Proposition 1.1.

Proof of Proposition 1.1. From (1.23) and the fact that

$$\pi\Big(q\Big(1,\frac{p}{q}\Big)\Big) = \pi(q,p) = \pi(0,0) \in \omega_0,$$

we infer that

$$\omega_0 + [0,q]\Big(1,\frac{p}{q}\Big) = \mathbb{T}^2.$$

We introduce the set

$$K := \Big\{s(L,0) + t\Big(1,\frac{p}{q}\Big);\ s\in[0,1],\ t\in[0,q]\Big\} \subset \mathbb{R}^2$$

and its ε-neighborhood

$$K_\varepsilon := \{(x,y)\in\mathbb{R}^2;\ \mathrm{dist}((x,y),K) < \varepsilon\} = K + B((0,0),\varepsilon).$$

Then it follows from the definitions of ω_0 and K_ε that

$$\mathbb{T}^2 = \omega_0 + [0,q]\Big(1,\frac{p}{q}\Big) = \pi(K_\varepsilon).$$

This yields $1 = \lambda([0,1]^2) \le \lambda(K_\varepsilon)$, where λ denotes Lebesgue measure. Elementary computations give

$$\lambda(K_\varepsilon) = Lp + 2\varepsilon L + 2\varepsilon\sqrt{p^2+q^2} + \pi\varepsilon^2,$$

and (1.24) follows. □

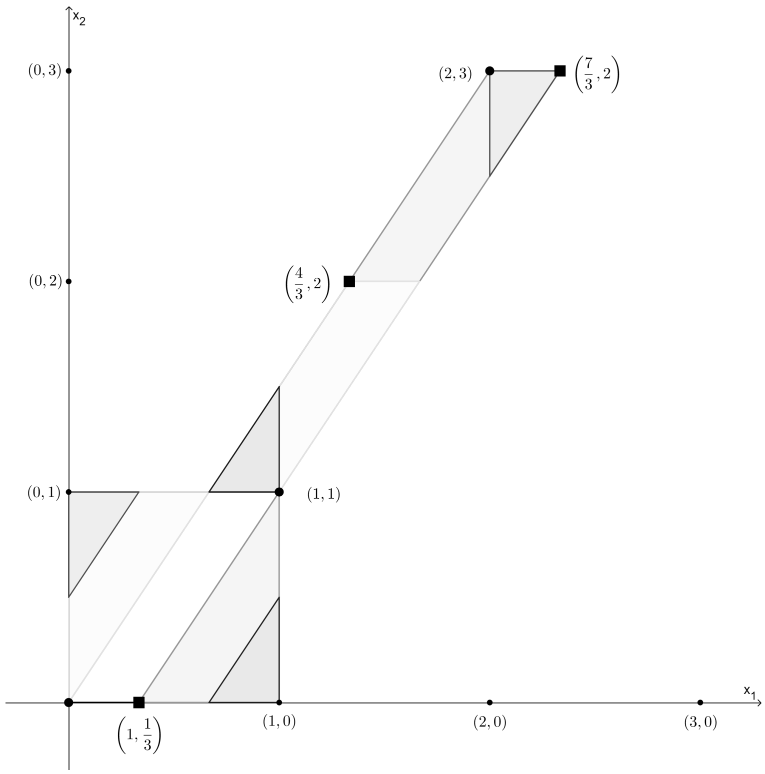

FIG. 1. *The domain D for $(p, q) = (3, 2)$.*

We introduce some notation. For any real number R, let $[R]$ (resp. $\{R\}$) denote its *integral part* (resp., its *fractional part*); that is, $[R] := \sup\{n \in \mathbb{Z};\ n \le R\} \in \mathbb{Z}$ and $\{R\} := R - [R] \in [0, 1)$.

The following result shows that to define the function ψ_0 as a $\mathbb{Z}^2$-periodic function on $\mathbb{R}^2$, it is sufficient to restrict oneself to the domain D. (See Figure 1.)

PROPOSITION 2.2. *Let $p, q \in \mathbb{N}^*$ be relatively prime. Let*

$$D := \left\{ s\left(\frac{1}{p}, 0\right) + t\left(1, \frac{p}{q}\right),\ s \in [0, 1),\ t \in [0, q) \right\}.$$

Then for any $z \in \mathbb{R}^2$ there exists a unique pair $(x, y) \in D \times \mathbb{Z}^2$ such that $z = x + y$. Accordingly, $\pi(D) = \mathbb{T}^2$.

Proof. It is sufficient to show that the function $f : D \to [0, 1)^2$ defined by $f(x_1, x_2) = (\{x_1\}, \{x_2\})$ is bijective. Indeed, in such a case, we would have that for any given $(z_1, z_2) \in \mathbb{R}^2$, there would exist a unique $x = (x_1, x_2) \in D$ such that $x - ([x_1], [x_2]) = z - ([z_1], [z_2]) \in [0, 1)^2$. Therefore, setting $y := ([z_1], [z_2]) - ([x_1], [x_2]) \in \mathbb{Z}^2$, we would get the desired result.

The fact that $f : D \to [0, 1)^2$ is bijective means that the system

$$\left\{ t + \frac{1}{p} s \right\} = z_1, \tag{2.3}$$

$$\left\{ \frac{p}{q} t \right\} = z_2 \tag{2.4}$$

admits a unique solution $(s,t) \in [0,1) \times [0,q)$ for any $(z_1,z_2) \in [0,1)^2$.

Let us first show that $f : D \to [0,1)^2$ is one-to-one. Let $(s_1,s_2) \in [0,1)^2$ and $(t_1,t_2) \in [0,q)^2$ be such that

$$f\Big(t_1 + \frac{s_1}{p}, \frac{t_1 p}{q}\Big) = f\Big(t_2 + \frac{s_2}{p}, \frac{t_2 p}{q}\Big);$$

that is, $\{t_1 + s_1/p\} = \{t_2 + s_2/p\}$ and $\{t_1 p/q\} = \{t_2 p/q\}$. Then, setting $k_1 := [t_1 + s_1/p] - [t_2 + s_2/p] \in \mathbb{Z}$ and $k_2 := [t_1 p/q] - [t_2 p/q] \in \mathbb{Z}$, we obtain that

$$\Big(t_1 + \frac{s_1}{p}\Big) - \Big(t_2 + \frac{s_2}{p}\Big) = \Big[t_1 + \frac{s_1}{p}\Big] + \Big\{t_1 + \frac{s_1}{p}\Big\} - \Big(\Big[t_2 + \frac{s_2}{p}\Big] + \Big\{t_2 + \frac{s_2}{p}\Big\}\Big) = k_1,$$
$$t_1\frac{p}{q} - t_2\frac{p}{q} = \Big[t_1\frac{p}{q}\Big] + \Big\{t_1\frac{p}{q}\Big\} - \Big(\Big[t_2\frac{p}{q}\Big] + \Big\{t_2\frac{p}{q}\Big\}\Big) = k_2.$$

It follows that

$$s_1 - s_2 = pk_1 - qk_2, \tag{2.5}$$
$$(t_1 - t_2)p = qk_2. \tag{2.6}$$

Since $s_1 - s_2 \in (-1,1)$, the only possible value for $pk_1 - qk_2 \in \mathbb{Z}$ is zero. Thus $pk_1 = qk_2$ and $s_1 = s_2$. Since p and q are relatively prime, we infer that we can write $k_2 = lp$ for some $l \in \mathbb{Z}$. It follows that $(t_1 - t_2)p = lpq$, and hence $t_1 - t_2 = lq$. But $t_1 - t_2 \in (-q,q)$, so we infer that $t_1 = t_2$.

Let us prove that $f : D \to [0,1)^2$ is onto. Given $(z_1,z_2) \in [0,1)^2$, we aim to find $s \in [0,1)$ and $t \in [0,q)$ such that system (2.3)–(2.4) is satisfied. Note that (2.3)–(2.4) is equivalent to the existence of some numbers $k_1, k_2 \in \mathbb{Z}$ such that

$$t + \frac{s}{p} = z_1 + k_1, \tag{2.7}$$
$$t\frac{p}{q} = z_2 + k_2. \tag{2.8}$$

Eliminating t from (2.7), we arrive at the equivalent system

$$s = (pz_1 - qz_2) + (pk_1 - qk_2), \tag{2.9}$$
$$pt = qz_2 + qk_2. \tag{2.10}$$

Since $pk_1 - qk_2 \in \mathbb{Z}$, we have to pick $s = \{pz_1 - qz_2\}$. Next, (2.9) will be satisfied if and only if

$$pk_1 - qk_2 = -[pz_1 - qz_2]. \tag{2.11}$$

Since p and q are relatively prime, there exists by Bézout's identity a pair $(a,b) \in \mathbb{Z}^2$ such that $ap + bq = 1$. Let $l := -[pz_1 - qz_2] \in \mathbb{Z}$ and

$$k_1 := al + rq, \quad k_2 := -bl + rp,$$

where $r \in \mathbb{Z}$ remains to be defined. Then $k_1, k_2 \in \mathbb{Z}$ and (2.11) is satisfied. It remains to fulfill (2.10), which can be rewritten $t/q = (z_2 - bl)/p + r$, with the constraint $t \in [0,q)$. Picking $r := -[(z_2 - bl)/p]$ and $t := q((z_2 - bl)/p + r) \in [0,q)$, we infer that (2.10) is also satisfied. The proof of Proposition 2.2 is complete. ∎

Proposition 2.3. *Let $c = (1, p/q)$, where $p, q \in \mathbb{N}^*$ are relatively prime. Let $\omega_0 \subset \mathbb{T}^2$ be a nonempty open set such that $\pi([0, 1/p] \times \{0\}) \subset \omega_0$, and let $\tau \in (0, 1]$ be given. Then there exists a nonnegative function $\psi_0 \in C^\infty(\mathbb{T}^2)$ such that*

$$2\tau \le c \cdot \nabla\psi_0 \le 2(q + \tau) \quad in \quad \mathbb{T}^2 \backslash \omega_0, \tag{2.12}$$

$$c \cdot \nabla(c \cdot \nabla\psi_0) = 2 \quad in \quad \mathbb{T}^2 \backslash \omega_0. \tag{2.13}$$

Proof. Our goal is to construct a suitable function ψ_0 in D and then to extend it to $\mathbb{R}^2$ as a smooth $\mathbb{Z}^2$–periodic function. Therefore, it is important to determine what pairs of points on ∂D are identified when viewed as points in $\mathbb{T}^2$, that is, modulo $\mathbb{Z}^2$. We first focus on the point $(1/p, 0) \in \partial D$ and determine all the points on ∂D that can be identified with it. (See Figure 2.)

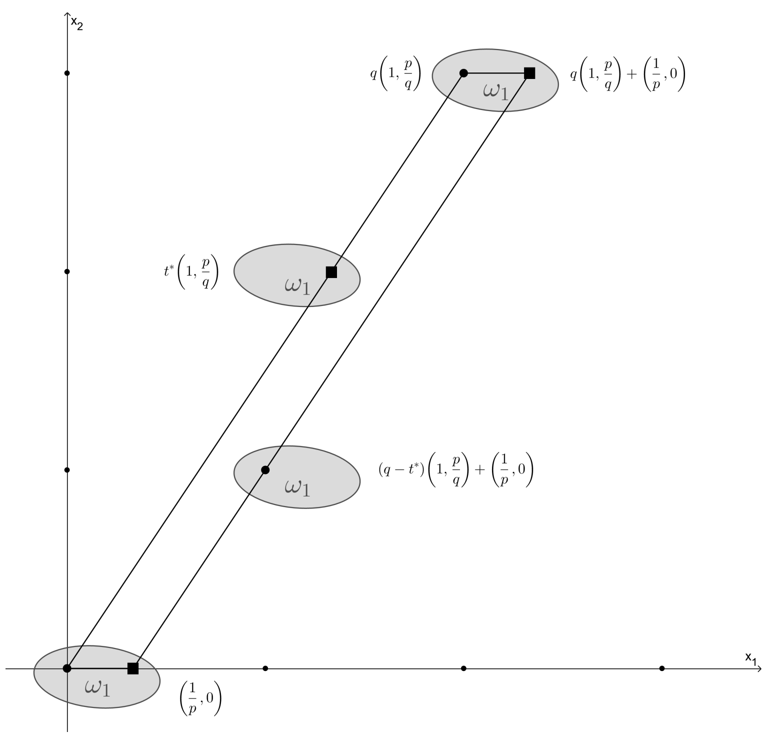

Fig. 2. *The domain D and the open set ω_1 for $(p, q) = (3, 2)$. Note that $t^* = 4/3$.*

Lemma 2.1. *There exists a unique number $t^* \in [0, q)$ such that $t^*(1, p/q) - (1/p, 0) \in \mathbb{Z}^2$.*

Proof of Lemma 2.1. Since p and q are relatively prime, by Bézout's identity there exists a pair $(a, b) \in \mathbb{Z}^2$ such that $ap+bq = 1$. Since $(a-qm)p+(b+mp)q = 1$ for any $m \in \mathbb{Z}$, we may assume without loss of generality that $b \in [0, p)$. Set $t^* = bq/p \in [0, q)$. Then $t^* - 1/p = -a \in \mathbb{Z}$ and $t^*p/q = b \in \mathbb{Z}$, so that $t^*(1, p/q) - (1/p, 0) \in \mathbb{Z}^2$. On the other hand, it follows from Proposition 2.2 that there exists a unique pair $(x, y) \in D \times \mathbb{Z}^2$ such that $(1/p, 0) = x+y$. Decomposing x as $x = s(1/p, 0)+t(1, p/q)$, we infer that there exists a unique pair $(s, t) \in [0, 1) \times [0, q)$ such that

$$(1/p, 0) - [s(1/p, 0) + t(1, p/q)] \in \mathbb{Z}^2.$$

This yields $(s,t)=(0,t^*)$ and the uniqueness of t^*. ∎

Let us go back to the proof of Proposition 2.3. We start with a claim, which is a direct consequence of Lemma 2.1.

CLAIM 1. For every $t\in[0,q-t^*]$, $(t+t^*)(1,p/q)\in\partial D$ is identified in $\mathbb{T}^2$ with $t(1,p/q)+(1/p,0)\in\partial D$. For every $t\in[q-t^*,q]$, $(t-(q-t^*))(1,p/q)\in\partial D$ is identified with $t(1,p/q)+(1/p,0)\in\partial D$.

Since $\pi([0,1/p]\times\{0\})\subset\omega_0$, we can pick two open sets ω_1,ω_2 in $\mathbb{T}^2$ such that $\pi([0,1/p]\times\{0\})\subset\omega_1$ and $\overline{\omega_1}\subset\omega_2\subset\overline{\omega_2}\subset\omega_0$. Since, by Lemma 2.1, $(q-t^*)(1,p/q)+(1/p,0)\in\mathbb{Z}^2$ is identified with $(0,0)$, and since the open set ω_1 contains $(0,0)$, we can select a number $\varepsilon\in(0,1)$ such that for every $s\in[1-\varepsilon,1]$ we have that $(q-t^*)(1,p/q)+s(1/p,0)\in\omega_1$.

We need to introduce two cut-off functions. Pick $\chi\in C^\infty(\mathbb{T}^2)$ with $0\le\chi\le1$ such that

$$\chi(x)=\begin{cases}1 & \text{if } x\in\mathbb{T}^2\setminus\omega_2,\\ 0 & \text{if } x\in\overline{\omega_1},\end{cases}$$

and $\eta\in C^\infty([0,1])$ with $0\le\eta\le1$ and such that

$$\eta(s)=\begin{cases}1 & \text{if } s\le 1-\varepsilon,\\ 0 & \text{if } s\ge 1-\frac{\varepsilon}{2}.\end{cases}$$

Introduce the functions $\psi_1,\psi_2:[0,q]\to\mathbb{R}$ defined by

$$\psi_1(t)=(t+\tau)^2,\quad t\in[0,q], \tag{2.14}$$

$$\psi_2(t)=\begin{cases}(t+t^*+\tau)^2 & \text{if } t\in[0,q-t^*],\\ (t+t^*-q+\tau)^2 & \text{if } t\in(q-t^*,q].\end{cases} \tag{2.15}$$

Note that

$$\psi_2(t)=\begin{cases}\psi_1(t+t^*) & \text{if } t\in[0,q-t^*),\\ \psi_1(t-(q-t^*)) & \text{if } t\in(q-t^*,q].\end{cases}$$

Finally, taking into account the fact that any point $x\in D$ can be written in a unique way as $x=t(1,p/q)+s(1/p,0)$ with $(s,t)\in[0,1)\times[0,q)$, we define the function $\psi_0:D\to\mathbb{R}_+$ by

$$\begin{aligned}\psi_0(t(1,p/q)&+s(1/p,0))\\ &=[\eta(s)\psi_1(t)+(1-\eta(s))\psi_2(t)]\,\chi(t(1,p/q)+s(1/p,0)),\quad (s,t)\\ &\in[0,1)\times[0,q).\end{aligned}$$

Using Proposition 2.2, we can extend ψ_0 to $\mathbb{R}^2=D+\mathbb{Z}^2$ by setting

$$\psi_0(x+y)=\psi_0(x)\quad\forall x\in D,\ \forall y\in\mathbb{Z}^2.$$

The function ψ_0 being $\mathbb{Z}^2$-periodic, it can be viewed as a map from $\mathbb{T}^2$ to $\mathbb{R}$. It turns out that ψ_0 is smooth.

CLAIM 2. $\psi_0\in C^\infty(\mathbb{T}^2)$.

Indeed, the function ψ_0 is smooth:

(i) in the interior of $\pi(D)$, namely in the set $\{\pi\big(t(1,p/q)+s(1/p,0)\big);\ (s,t)\in(0,1)\times(0,q)\}$;

(ii) in a neighborhood of the segment $[0,1/p]\times\{0\}$ thanks to the definitions of ω_1 and χ;

(iii) in a neighborhood of the segment $\{(q-t^*)(1,p/q)+s(1/p,0);s\in[1-\varepsilon,1]\}$ thanks to the definition of ω_1, χ and η (even if the function ψ_2 is discontinuous at $t = q - t^*$); (iv) in a neighborhood of the segments $\{t(1, p/q); t \in [0, q]\}$ and $\{(1/p, 0)+t(1, p/q); t \in [0, q]\}$ thanks to Claim 1 and to the definitions of ψ_1 and ψ_2.

It remains to check that ψ_0 fulfills (2.12) and (2.13). But we notice that for $x = s(1/p, 0) + t(1, p/q) \in \mathbb{T}^2 \setminus \overline{\omega_2}$, $\chi(x) = 1$ and

$$c \cdot \nabla\psi_0(x) = \frac{\partial}{\partial t}[\psi_0(s(1/p, 0) + t(1, p/q))] = \eta(s)\psi_1'(t) + (1 - \eta(s))\psi_2'(t) \in [2\tau, 2(q + \tau)],$$

$$c \cdot \nabla(c \cdot \nabla\psi_0)(x) = \frac{\partial^2}{\partial t^2}[\psi_0(s(1/p, 0) + t(1, p/q))] = \eta(s)\psi_1''(t) + (1 - \eta(s))\psi_2''(t) = 2.$$

Noting that $\mathbb{T}^2 \setminus \omega_0 \subset \mathbb{T}^2 \setminus \overline{\omega_2}$, we infer that (2.12)–(2.13) are fulfilled. The proof of Proposition 2.3 is complete. ☐

(iii) $c = (1, \zeta)$**,** $\zeta \in \mathbb{R}_+ \setminus \mathbb{Q}$**.**

Before performing the construction of the function ψ_0, we first prove Proposition 1.2.

Proof of Proposition 1.2. If there does not exist $T > 0$ such that $\omega_0 + [0, T]c = \mathbb{T}^2$, then for any $n \in \mathbb{N}^*$ we can pick some $y_n \in \mathbb{T}^2$ such that

$$y_n \notin \omega_0 + [0, n]c. \tag{2.16}$$

As $\mathbb{T}^2$ is compact, we can extract a subsequence $y_{n_k} \to y$ for some $y \in \mathbb{T}^2$. Pick $x \in \mathbb{T}^2$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset \omega_0$, and let $z = (z_1, z_2) = y - x$. As the subgroup $\{\zeta p + q;\ p, q \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ (for $\zeta \notin \mathbb{Q}$), we infer the existence of a pair $(p, q) \in \mathbb{Z}^2$ such that $|z_2 - \zeta z_1 - (\zeta p + q)| < \varepsilon/4$. As $\zeta \in \mathbb{R}_+ \setminus \mathbb{Q} = \mathcal{I}_{1/\sqrt{5}}$, one may find a pair $(p', q') \in \mathbb{N}^2$ (with q' as large as desired) such that $|\zeta - \frac{p'}{q'}| \le (\sqrt{5}q'^2)^{-1}$. In particular, we can impose both conditions $q' \ge |z_1 + p| + 1$ and $(\sqrt{5}q')^{-1} < \varepsilon/4$. Set $t = z_1 + p + q' > 0$. Then

$$\begin{aligned} |(z_1, z_2) - t(1, \zeta) - (p + q', q - p')| &= |z_2 - (z_1 + p + q')\zeta - q + p'| \\ &\le |z_2 - \zeta z_1 - (\zeta p + q)| + |p' - q'\zeta| \\ &\le \frac{\varepsilon}{2}. \end{aligned}$$

It follows that for n large enough, $|y_n - x - tc - (p + q', q - p')| < \varepsilon$, that is, $y_n \in B(x, \varepsilon) + tc$, contradicting the assumption. ☐

Let us proceed with the construction of the function ψ_0 when $c = (1, \zeta)$ with $\zeta \in \mathbb{R} \setminus \mathbb{Q}_+$. The idea is to use the function ψ_0 constructed above for $\hat{c} = (1, \frac{p}{q})$ when p/q is a "good" rational approximation of ζ, namely satisfying (1.26). Note that in this case $|q\zeta - p| \le \delta/q$, so that

$$p \ge q\zeta - \frac{\delta}{q} \ge \frac{1}{2}\zeta q \tag{2.17}$$

provided that $A \gg 1$. (Recall that $q \ge A$.) Moreover,

$$\left|\frac{q}{p}\zeta - 1\right| \le \frac{\delta}{pq} \le \frac{2\delta}{\zeta q^2} \tag{2.18}$$

provided that $A \gg 1$.

PROPOSITION 2.4. *Assume that $c = (1, \zeta)$, where $\zeta \in \mathcal{I}_\delta$ for some $\delta > 0$. Let ω_0 be an open neighborhood of 0. Pick $A > 0$ and a pair (p, q) with $q \geq A$ such that* (1.22), (1.26), (2.17), *and* (2.18) *are satisfied, and such that*

$$\frac{1}{p} < dist(0, \mathbb{T}^2 \setminus \omega_0). \tag{2.19}$$

Let $\hat{c} = (1, \frac{p}{q})$. Then there exists a function $\psi_0 \in C^\infty(\mathbb{T}^2)$ such that

$$\nabla\psi_0(x) \neq 0 \quad in \quad \mathbb{T}^2 \backslash \omega_0, \tag{2.20}$$

$$|(c - \hat{c}) \cdot \nabla\psi_0| \leq 8\delta q + f_1(\delta, \zeta) \quad in \quad \mathbb{T}^2 \backslash \omega_0, \tag{2.21}$$

$$|c \cdot \nabla(c \cdot \nabla\psi_0) - 2| \leq f_2(\delta, \zeta) \quad in \quad \mathbb{T}^2 \backslash \omega_0, \tag{2.22}$$

where

$$f_1(\delta, \zeta) := \Big(\frac{8}{\zeta} + 16\Big)\delta, \tag{2.23}$$

$$f_2(\delta, \zeta) := \Big(32 + \frac{8}{\zeta}\Big)\delta + \Big(549 + \frac{64}{\zeta} + \frac{8}{\zeta^2}\Big)\delta^2. \tag{2.24}$$

Proof. It follows from (2.19) that $[-\frac{1}{p}, \frac{1}{p}] \times \{0\} \subset \omega_0$. We pick two open sets ω_1, ω_2 with

$$\Big[-\frac{1}{p}, \frac{1}{p}\Big] \times \{0\} \subset \omega_1 \subset \overline{\omega_1} \subset \omega_2 \subset \overline{\omega_2} \subset \omega_0.$$

Let the functions χ, η, ψ_1, ψ_2, and ψ_0 be as in the proof of Proposition 2.3 for $\hat{c} = (1, \frac{p}{q})$ and $\varepsilon = 1/2$. Note that it is easy to construct a function η as above with the following explicit bounds:

$$\|\eta'\|_{L^\infty(\mathbb{R})} \leq 2^3, \quad \|\eta''\|_{L^\infty(\mathbb{R})} \leq 2^7 \left(\int_{\mathbb{R}} \exp\Big(1 - \frac{1}{1 - y^2}\Big)\, dy\right)^{-1} \leq 183.$$

Clearly, (2.20) follows at once from (2.12). Set $\hat{d} = (\frac{1}{p}, 0)$. Then

$$c = (1, \zeta) = \frac{q}{p}\zeta\hat{c} + (p - q\zeta)\hat{d}. \tag{2.25}$$

From the definition of ψ_0, (2.17), (2.18), and (2.25), we obtain that for $x = s\hat{d} + t\hat{c} \in \mathbb{T}^2 \setminus \overline{\omega_2}$ (where $s \in [0, 1)$, $t \in [0, q)$),

$$\begin{aligned}(c - \hat{c}) \cdot \nabla\psi_0(x) &= \Big(\frac{q}{p}\zeta - 1\Big)\hat{c} \cdot \nabla\psi_0(x) + (p - q\zeta)\hat{d} \cdot \nabla\psi_0(x) \\ &= \Big(\frac{q}{p}\zeta - 1\Big)\frac{\partial}{\partial t}[\psi_0(s\hat{d} + t\hat{c})] + (p - q\zeta)\frac{\partial}{\partial s}[\psi_0(s\hat{d} + t\hat{c})] \\ &= \Big(\frac{q}{p}\zeta - 1\Big)\big(\eta(s)\psi_1'(t) + (1 - \eta(s))\psi_2'(t)\big) \\ &\quad + (p - q\zeta)\eta'(s)\big(\psi_1(t) - \psi_2(t)\big).\end{aligned}$$

Recall that for $i = 1, 2$,

$$\psi_i(t) \in [\tau^2, (q + \tau)^2], \quad \psi_i'(t) \in [2\tau, 2(q + \tau)], \quad \psi_i''(t) = 2. \qquad \square$$

It follows that

$$
\begin{aligned}
|(c-\hat{c})\cdot\nabla\psi_0(x)| &\le |\frac{q}{p}\zeta - 1|\cdot 2(q+\tau) + |p-q\zeta|\cdot 8\cdot(q^2+2q\tau)\\
&\le \frac{2\delta}{\zeta q^2}\cdot 2(q+\tau) + \frac{8\delta}{q}(q^2+2q\tau)\\
&\le \delta\Big(8q+\frac{8}{\zeta}+16\Big)
\end{aligned}
$$

for $q\ge 1$ and $\tau\le 1$. We can write

$$
\begin{aligned}
c\cdot\nabla(c\cdot\nabla\psi_0)(x) - \hat{c}\cdot\nabla(\hat{c}\cdot\nabla\psi_0) &= c\cdot\nabla((c-\hat{c})\cdot\nabla\psi_0)(x) + (c-\hat{c})\cdot\nabla(\hat{c}\cdot\nabla\psi_0)(x)\\
&=: I_1 + I_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_1 &= \frac{q}{p}\zeta\frac{\partial}{\partial t}\Big([(c-\hat{c})\cdot\nabla\psi_0](s\hat{d}+t\hat{c})\Big) + (p-q\zeta)\frac{\partial}{\partial s}\Big([(c-\hat{c})\cdot\nabla\psi_0](s\hat{d}+t\hat{c})\Big)\\
&= \frac{q}{p}\zeta\left[\Big(\frac{q}{p}\zeta-1\Big)(\eta(s)\psi_1''(t)+(1-\eta(s))\psi_2''(t)) + (p-q\zeta)\eta'(s)(\psi_1'(t)-\psi_2'(t))\right]\\
&\quad + (p-q\zeta)\left[\Big(\frac{q}{p}\zeta-1\Big)\eta'(s)(\psi_1'(t)-\psi_2'(t)) + (p-q\zeta)\eta''(s)(\psi_1(t)-\psi_2(t))\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
|I_1| &\le \Big|\frac{q}{p}\zeta\Big|\left[\Big|\frac{q}{p}\zeta-1\Big|\cdot 2 + |p-q\zeta|\cdot 2^3\cdot 2q\right]\\
&\qquad + |p-q\zeta|\left[\Big|\frac{q}{p}\zeta-1\Big|\cdot 8\cdot 2q + |p-q\zeta|\cdot 183\cdot(q^2+2\tau q)\right]\\
&\le \left(1+\frac{2\delta}{\zeta q^2}\right)\left(\frac{2\delta}{\zeta q^2}\cdot 2 + \frac{\delta}{q}\cdot 16q\right)\\
&\qquad + \frac{\delta}{q}\left(\frac{2\delta}{\zeta q^2}\cdot 16q + \frac{\delta}{q}\cdot 183\cdot(q^2+2\tau q)\right)\\
&\le 16\delta + 183\delta^2 + \frac{32\delta^2}{\zeta q^2} + \frac{16\delta^2}{q} + \left(\frac{32\delta^2}{\zeta}+\frac{4\delta}{\zeta}\right)\frac{1}{q^2} + \frac{8\delta^2}{\zeta^2 q^4},
\end{aligned}
$$

where we used the fact that $\tau\in[0,1]$. On the other hand, we have that

$$
\begin{aligned}
I_2 &= \Big(\frac{q}{p}\zeta-1\Big)\hat{c}\cdot\nabla(\hat{c}\cdot\nabla\psi_0) + (p-q\zeta)\hat{d}\cdot\nabla(\hat{c}\cdot\nabla\psi_0)\\
&= \Big(\frac{q}{p}\zeta-1\Big)\frac{\partial^2}{\partial t^2}[\psi_0(s\hat{d}+t\hat{c})] + (p-q\zeta)\cdot\frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}[\psi_0(s\hat{d}+t\hat{c})]\right)\\
&= \Big(\frac{q}{p}\zeta-1\Big)(\eta(s)\psi_1''(t)+(1-\eta(s))\psi_2''(t)) + (p-q\zeta)\eta'(s)(\psi_1'(t)-\psi_2'(t)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|I_2| &\le \Big|\frac{q}{p}\zeta-1\Big|\cdot 2 + |p-q\zeta|\cdot 8\cdot 2q\\
&\le \frac{2\delta}{\zeta q^2}\cdot 2 + \frac{\delta}{q}\cdot 16q = 16\delta + \frac{4\delta}{\zeta q^2}.
\end{aligned}
$$

We infer that

$$\begin{aligned}|c\cdot\nabla(c\cdot\nabla\psi_0)-2| &= |c\cdot\nabla(c\cdot\nabla\psi_0)-\hat{c}\cdot\nabla(\hat{c}\cdot\nabla\psi_0)|\\ &\le |I_1|+|I_2|\\ &\le 32\delta+183\delta^2\\ &\qquad +366\frac{\delta^2}{q}+\Big(\frac{8\delta}{\zeta}+\frac{64\delta^2}{\zeta}\Big)\frac{1}{q^2}+\frac{8\delta^2}{\zeta^2q^4}\\ &\le \Big(32+\frac{8}{\zeta}\Big)\delta+\Big(549+\frac{64}{\zeta}+\frac{8}{\zeta^2}\Big)\delta^2.\end{aligned}$$

3. Carleman estimates.

In this section, we derive a Carleman estimate for a transport equation and a Carleman estimate for a heat equation with the *same* weights.

Let $\psi_0\in C^\infty(\mathbb{T}^2)$, $K,\widetilde{K}\in(0,+\infty)$, and $t_0\in[0,T]$ (to be chosen later on), and set

$$\psi(x,t)=\psi_0(x)-K(t-t_0)^2+\widetilde{K},\quad (x,t)\in\mathbb{T}^2\times[0,T]. \tag{3.1}$$

By picking $\widetilde{K}$ large enough, we can assume that

$$\psi(x,t)>\frac{3}{4}\|\psi\|_{L^\infty(\mathbb{T}^2\times(0,T))}\qquad\forall x\in\mathbb{T}^2,\ \forall t\in[0,T]. \tag{3.2}$$

For given $\sigma\in(0,\min(1,T/2))$, we introduce some function $g\in C^\infty(0,T)$ such that

$$g(t)=\begin{cases}\frac{1}{t} & \text{for } 0<t<\frac{\sigma}{2},\\ \text{strictly decreasing} & \text{for } 0<t\le\sigma,\\ 1 & \text{for } \sigma\le t\le\frac{T}{2},\\ g(T-t) & \text{for } \frac{T}{2}<t\le T.\end{cases} \tag{3.3}$$

We define the weights

$$\varphi(x,t):=g(t)\left(e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty(\mathbb{T}^2\times(0,T))}}-e^{\lambda\psi(x,t)}\right), \tag{3.4}$$

$$\theta(x,t):=g(t)e^{\lambda\psi(x,t)}, \tag{3.5}$$

where $\lambda>0$ is a parameter.

The Carleman estimate for the transport equation is given in the following proposition.

PROPOSITION 3.1. *Let $c=(1,\zeta)$ with $\zeta\in\mathbb{R}_+$. Let $T>0$, $\sigma\in(0,\min(1,T/2))$ and $K,\tilde{K}>0$. Assume that the function $\psi_0\in C^\infty(\mathbb{T}^2)$ satisfies for some nonempty open set $\omega_0\subset\mathbb{T}^2$ and some number $\hat{\delta}>0$,*

$$\nabla\psi_0(x)\neq 0\quad\forall x\in\mathbb{T}^2\setminus\omega_0, \tag{3.6}$$

$$-2K(t-t_0)-c\cdot\nabla\psi_0(x)\ge 0\quad\forall x\in\mathbb{T}^2\setminus\omega_0,\ \forall t\in(0,\sigma), \tag{3.7}$$

$$-2K(t-t_0)-c\cdot\nabla\psi_0(x)\le 0\quad\forall x\in\mathbb{T}^2\setminus\omega_0,\ \forall t\in(T-\sigma,T), \tag{3.8}$$

$$c\cdot\nabla(c\cdot\nabla\psi_0)(x)-2K\ge\hat{\delta}\quad\forall x\in\mathbb{T}^2\setminus\omega_0. \tag{3.9}$$

Let the functions ψ, g, φ, and θ be as in (3.1)–(3.5). Then there exist some constants $\lambda_0\ge 1$, $s_0\ge 1$, and $C_0>0$ such that for all $\lambda\ge\lambda_0$, all $s\ge s_0$, and all $w\in$

$L^2(\mathbb{T}^2\times(0,T))$ *with* $-w_t + c\cdot\nabla w \in L^2(\mathbb{T}^2\times(0,T))$, *we have*

$$\int_0^T\int_{\mathbb{T}^2}\lambda s\theta|w|^2e^{-2s\varphi}dxdt \tag{3.10}$$
$$\leq C_0\left(\int_0^T\int_{\mathbb{T}^2}|-w_t+c\cdot\nabla w|^2e^{-2s\varphi}dxdt+\int_0^T\int_{\omega_0}\lambda s\theta\Big(s\frac{|g'|}{g}\varphi+1\Big)|w|^2e^{-2s\varphi}dxdt\right).$$

Proof. Assume first that $w\in H^1(\mathbb{T}^2\times(0,T))$. Let $z=e^{-s\varphi}w$ and $Pw=-w_t+c\cdot\nabla w$. Then

$$\begin{aligned}e^{-s\varphi}Pw &:= e^{-s\varphi}P(e^{s\varphi}z) = -(s\varphi_t z+z_t)+c\cdot(sz\nabla\varphi+\nabla z)\\ &= P_1\,z+P_2\,z\end{aligned}$$

with

$$\begin{aligned}P_1z &:= -s\varphi_t z - z_t + c\cdot\nabla z,\\ P_2z &:= s(c\cdot\nabla\varphi)z.\end{aligned}$$

It follows that
$$\|e^{-s\varphi}Pw\|^2=\|P_1z\|^2+\|P_2z\|^2+2(P_1z,P_2z),$$
where $(f,g)=\int_0^T\int_{\mathbb{T}^2}fg\,dxdt$, $\|f\|^2=(f,f)$. In what follows, $\int_0^T\int_{\mathbb{T}^2}f(x,t)dxdt$ is denoted $\iint f$ for the sake of shortness.

The idea is to expand $(P_1\,z,P_2\,z)$ and next to use the (nonnegative) terms $\|P_1\,z\|^2$ and $\|P_2\,z\|^2$ to balance the "bad terms" in $(P_1\,z,P_2\,z)$. Using integration by parts, we obtain

$$\begin{aligned}(P_1z,P_2z) &= -s^2\iint(c\cdot\nabla\varphi)\varphi_t z^2 - s\iint(c\cdot\nabla\varphi)zz_t+\frac{s}{2}\iint(c\cdot\nabla\varphi)c\cdot\nabla z^2\\ &= -s^2\iint(c\cdot\nabla\varphi)\varphi_t z^2+\frac{s}{2}\iint(c\cdot\nabla\varphi_t)z^2-\frac{s}{2}\iint c\cdot\nabla(c\cdot\nabla\varphi)z^2.\end{aligned}$$

Easy calculations show that

$$\begin{aligned}\nabla\varphi &= -\lambda\theta\nabla\psi,\\ \varphi_t &= \frac{g'}{g}\varphi-\lambda\theta\psi_t,\\ \nabla\varphi_t &= -\lambda\frac{g'}{g}\theta\nabla\psi-\lambda^2\theta\psi_t\nabla\psi,\\ \nabla(c\cdot\nabla\varphi) &= -\lambda(c\cdot\nabla\psi)\nabla\theta-\lambda\theta\nabla(c\cdot\nabla\psi),\\ c\cdot\nabla(c\cdot\nabla\varphi) &= -\lambda^2(c\cdot\nabla\psi)^2\theta-\lambda\theta c\cdot\nabla(c\cdot\nabla\psi),\\ \nabla\psi_t &= 0.\end{aligned}$$

We infer that

$$\begin{aligned}(P_1z,P_2z)=\iint\Big[&s^2\lambda(c\cdot\nabla\psi)\theta\Big(\frac{g'}{g}\varphi-\lambda\theta\psi_t\Big)-\frac{s}{2}\Big(\lambda\frac{g'}{g}\theta c\cdot\nabla\psi+\lambda^2\theta\psi_t c\cdot\nabla\psi\Big)\\ &+\frac{s}{2}\left(\lambda^2(c\cdot\nabla\psi)^2\theta+\lambda\theta c\cdot\nabla(c\cdot\nabla\psi)\right)\Big]\,z^2.\end{aligned}$$

Next,

$$\|P_2 z\|^2 = s^2\lambda^2 \iint (c\cdot\nabla\psi)^2\theta^2 z^2$$

and

$$0 \le \|P_1 z - s\lambda\psi_t\theta z\|^2 = \|P_1 z\|^2 + s^2\lambda^2\|\psi_t\theta z\|^2 - 2s\lambda(P_1 z, \psi_t\theta z).$$

But

$$\begin{aligned}(P_1 z, \psi_t\theta z) &= -s\iint \varphi_t\psi_t\theta z^2 - \iint \psi_t\theta z_t z + \iint (c\cdot\nabla z)\psi_t\theta z\\ &= -s\iint \Big(\frac{g'}{g}\varphi - \lambda\theta\psi_t\Big)\psi_t\theta z^2 + \iint \Big(\psi_{tt}\theta + \psi_t\frac{g'}{g}\theta + \lambda\theta\psi_t^2\Big)\frac{z^2}{2}\\ &\quad - \iint \lambda\psi_t(c\cdot\nabla\psi)\theta\frac{z^2}{2}\cdot\end{aligned}$$

Therefore,

$$\begin{aligned}2s\lambda(P_1 z, \psi_t\theta z) &= 2s^2\lambda^2\iint \psi_t^2\theta^2 z^2 - 2s^2\lambda\iint \frac{g'}{g}\varphi\psi_t\theta z^2\\ &\quad + s\lambda\iint\Big(\psi_{tt}\theta + \frac{g'}{g}\theta\psi_t + \lambda\theta\psi_t^2 - \lambda\psi_t(c\cdot\nabla\psi)\theta\Big)z^2\\ &\le \|P_1 z\|^2 + s^2\lambda^2\iint \psi_t^2\theta^2 z^2,\end{aligned}$$

and hence

$$\begin{aligned}&s^2\lambda^2\iint \psi_t^2\theta^2 z^2 - 2s^2\lambda\iint \frac{g'}{g}\varphi\psi_t\theta z^2\\ &\qquad + s\lambda\iint\Big(\psi_{tt}\theta + \frac{g'}{g}\theta\psi_t + \lambda\theta\psi_t(\psi_t - c\cdot\nabla\psi)\Big)z^2 \le \|P_1 z\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\|e^{-s\varphi}Pw\|^2 &= \|P_1 z\|^2 + \|P_2 z\|^2 + 2(P_1 z, P_2 z)\\ &\ge s^2\lambda^2\iint(\psi_t^2 + (c\cdot\nabla\psi)^2 - 2\psi_t c\cdot\nabla\psi)\theta^2 z^2\\ &\quad + 2s^2\lambda\iint \frac{g'}{g}\varphi\theta(-\psi_t + c\cdot\nabla\psi)z^2\\ &\quad + s\lambda^2\iint\big(-\psi_t c\cdot\nabla\psi + (c\cdot\nabla\psi)^2 + \psi_t(\psi_t - c\cdot\nabla\psi)\big)\theta z^2\\ &\quad + s\lambda\iint\Big(-\frac{g'}{g}c\cdot\nabla\psi + c\cdot\nabla(c\cdot\nabla\psi) + \psi_{tt} + \frac{g'}{g}\psi_t\Big)\theta z^2\\ &= s\lambda^2\iint(\psi_t - c\cdot\nabla\psi)^2\theta^2 z^2 + s\lambda^2\iint(\psi_t - c\cdot\nabla\psi)^2\theta z^2\\ &\quad + s\lambda\iint\Big(\frac{-g'}{g}\Big)(\psi_t - c\cdot\nabla\psi)\theta(2s\varphi - 1)z^2\\ &\quad + s\lambda\iint(\psi_{tt} + c\cdot\nabla(c\cdot\nabla\psi))\theta z^2.\end{aligned}$$

With the two first integral terms of the right-hand side being nonnegative, we arrive at

$$\|e^{-s\varphi}Pw\|^2 \geq s\lambda \iint \Big(\frac{-g'}{g}\Big)(\psi_t - c\cdot\nabla\psi)\theta(2s\varphi-1)z^2 + s\lambda\iint(\psi_{tt}+c\cdot\nabla(c\cdot\nabla\psi))\theta z^2.$$

Noticing that

$$\varphi(x,t) = g(t)e^{\lambda\psi(x,t)}\left(e^{\frac{3}{2}\lambda\|\psi\|_{L^\infty}-\lambda\psi(x,t)}-1\right) \geq e^{-KT^2}\frac{\lambda\|\psi\|_{L^\infty}}{2},$$

we have that $s\varphi \geq 1$ for $\lambda \geq \lambda_0 = 1$ and $s \geq s_0$ with s_0 large enough. Using (3.7)–(3.8) and the fact that

$$g'(t)\begin{cases} \leq 0 & \text{if } \ t\in(0,\sigma),\\ = 0 & \text{if } \ t\in[\sigma, T-\sigma],\\ \geq 0 & \text{if } \ t\in(T-\sigma,T),\end{cases}$$

we infer that for $\lambda \geq \lambda_0$ and $s \geq s_0$

$$s\lambda\iint\Big(\frac{-g'}{g}\Big)(\psi_t - c\cdot\nabla\psi)\theta(2s\varphi-1)z^2 \geq s\lambda\int_0^T\!\!\int_{\omega_0}\Big(\frac{-g'}{g}\Big)(\psi_t-c\cdot\nabla\psi)\theta(2s\varphi-1)z^2,$$

and hence

$$s\lambda\iint(\psi_{tt}+c\cdot\nabla(c\cdot\nabla\psi))\theta z^2 \leq \|e^{-s\varphi}Pw\|^2 + Cs^2\lambda\int_0^T\!\!\int_{\omega_0}\frac{|g'|}{g}\theta\varphi|z|^2.$$

Using (3.9), we obtain for $\lambda \geq \lambda_0$ and $s \geq s_0$

$$s\lambda\iint\theta\, z^2 \leq C\left(\|e^{-s\varphi}Pw\|^2 + s\lambda\int_0^T\!\!\int_{\omega_0}\Big(s\frac{|g'|}{g}\varphi+1\Big)\theta\, z^2\,dxdt\right). \tag{3.11}$$

Replacing z by $e^{-s\varphi}w$ results in (3.10), which is thus established when $w\in H^1(\mathbb{T}^2\times(0,T))$.

We claim that (3.10) is still true when w and $f = -w_t + c\cdot\nabla w$ are in $L^2(0,T,L^2(\mathbb{T}^2))$. Indeed, in that case $w\in C([0,T],L^2(\mathbb{T}^2))$, and if (w_T^n) and (f^n) are two sequences in $H^1(\mathbb{T}^2)$ and $L^2(0,T,H^1(\mathbb{T}^2))$, respectively, such that

$$\begin{aligned} w_T^n &\to w(T) && \text{in } L^2(\mathbb{T}^2),\\ f^n &\to f && \text{in } L^2(0,T,L^2(\mathbb{T}^2)),\end{aligned}$$

then the solution $w^n\in C([0,T],H^1(\mathbb{T}^2))$ of

$$\begin{aligned} -w_t^n + c\cdot\nabla w^n &= f^n,\\ w^n(T) &= w_T^n\end{aligned}$$

satisfies $w^n\in H^1(\mathbb{T}^2\times(0,T))$ and $w^n\to w$ in $C([0,T],L^2(\mathbb{T}^2))$, so that we can apply (3.10) to w^n and pass to the limit $n\to+\infty$ in (3.10). The proof of Proposition 3.1 is complete. ∎

3.1. Carleman estimate for the parabolic equation.

PROPOSITION 3.2. *Let $T, \sigma, K, \widetilde{K}, \omega_0, \psi_0, \psi, g, \varphi$, and θ be as in Proposition* 3.1. *Then there exist some constants $\lambda_1 \geq \lambda_0$, $s_1 \geq s_0$, and $C_1 > 0$ such that for all $\lambda \geq \lambda_1$, all $s \geq s_1$, and all $v \in C([0,T], L^2(\mathbb{T}^2))$ with $v_t + \Delta v \in L^2(0,T,L^2(\mathbb{T}^2))$, the following holds:*

$$\int_0^T\int_{\mathbb{T}^2} [(s\theta)^{-1}(|\Delta v|^2 + |v_t|^2) + \lambda^2(s\theta)|\nabla v|^2 + \lambda^4(s\theta)^3|v|^2]e^{-2s\varphi}dxdt \\ \leq C_1\left(\int_0^T\int_{\mathbb{T}^2} |v_t + \Delta v|^2 e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0} \lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt\right). \tag{3.12}$$

Proof. The proof, similar to those in [4, 7, 18], is omitted here for the sake of brevity. The interested reader is referred to the preprint [10] for a detailed proof. □

We are now in a position to establish the observability inequality for the adjoint system (1.35)–(1.38).

PROPOSITION 3.3. *Let $T, \sigma, K, \widetilde{K}, \omega_0, \psi_0, \psi, g, \varphi$, and θ be as in Proposition* 3.1. *Pick any open set ω with $\overline{\omega_0} \subset \omega$. Then there exist some constants $\lambda_2 \geq \lambda_1$, $s_2 \geq s_1$, and $C_2 > 0$ such that for all $\lambda \geq \lambda_2$, all $s \geq s_2$, and all $(v_T, w_T) \in [L^2(\mathbb{T}^2)]^2$, if (v,w) denotes the solution of* (1.35)–(1.38)*, then we have*

$$\int_0^T\int_{\mathbb{T}^2} [\lambda s\theta|w|^2 + (s\theta)^{-1}(|\Delta v|^2 + |v_t|^2) + \lambda^2(s\theta)|\nabla v|^2 + \lambda^4(s\theta)^3|v|^2]e^{-2s\varphi}dxdt \\ \leq C_2\int_0^T\int_\omega \lambda^8(s\theta)^7|w|^2e^{-2s\varphi}dxdt. \tag{3.13}$$

As a consequence, there is a constant $C_{obs} > 0$ independent of (v_T, w_T) such that

$$\int_{\mathbb{T}^2} \left(|v(x,0)|^2 + |w(x,0)|^2\right)dx \leq C_{obs}\int_0^T\int_\omega |w(x,t)|^2dxdt. \tag{3.14}$$

Proof. It is clear that $v \in C([0,T], L^2(\mathbb{T}^2)) \cap L^2(0,T,H^1(\mathbb{T}^2))$, and $w \in L^2(0,T, L^2(\mathbb{T}^2))$, so that by (1.35)–(1.36), $v_t + \Delta v \in L^2(0,T,L^2(\mathbb{T}^2))$ and $-w_t + c \cdot \nabla w \in$

$L^2(0,T,L^2(\mathbb{T}^2))$. It follows then from (3.10) and (3.12) that we have

$$\int_0^T\int_{\mathbb{T}^2} \big[\lambda s\theta|w|^2 e^{-2s\varphi}dxdt + (s\theta)^{-1}(|\Delta v|^2+|v_t|^2)+\lambda^2(s\theta)|\nabla v|^2 + \lambda^4(s\theta)^3|v|^2\big]e^{-2s\varphi}dxdt$$
$$\le C\left(\int_0^T\int_{\mathbb{T}^2}|-w_t+c\cdot\nabla w|^2e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0}\lambda s\theta|w|^2\Big(s\frac{|g'|}{g}\varphi+1\Big)e^{-2s\varphi}dxdt \right.$$
$$\left. + \int_0^T\int_{\mathbb{T}^2}|v_t+\Delta v|^2e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0}\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt\right)$$
$$\le C\left(\int_0^T\int_{\mathbb{T}^2}|v-w|^2e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0}\lambda s\theta|w|^2\Big(s\frac{|g'|}{g}\varphi+1\Big)e^{-2s\varphi}dxdt \right.$$
$$\left. + \int_0^T\int_{\mathbb{T}^2}|c\cdot\nabla v+\big(b(x+tc)-1\big)(v-w)|^2e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0}\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt\right),$$

where $C := \max(C_0, C_1)$. In what follows, C denotes a constant (which does not depend on s, λ, v_T, w_T) that may vary from line to line. Taking $s \ge s_2 \ge s_1$ and $\lambda \ge \lambda_2 \ge \lambda_1$ with s_2, λ_2 large enough, we can absorb the first and third integral terms in the right-hand side of the last equation, so that

$$\int_0^T\int_{\mathbb{T}^2} \big[\lambda s\theta|w|^2 e^{-2s\varphi} + (s\theta)^{-1}(|\Delta v|^2+|v_t|^2)+\lambda^2(s\theta)|\nabla v|^2 + \lambda^4(s\theta)^3|v|^2\big]e^{-2s\varphi}dxdt$$
$$\le C\left(\int_0^T\int_{\omega_0}\lambda s\theta|w|^2\big(s\frac{|g'|}{g}\varphi+1\big)e^{-2s\varphi}dxdt + \int_0^T\int_{\omega_0}\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt\right). \tag{3.15}$$

It remains to eliminate the last integral term. To do this, following [20], we derive local energy estimates. We introduce a cut-off function χ such that

$$\chi \in C_0^\infty(\omega), \tag{3.16}$$
$$0 \le \chi(x) \le 1, \quad x \in \mathbb{T}^2, \tag{3.17}$$
$$\chi(x) = 1, \quad x \in \omega_0. \tag{3.18}$$

We have that

$$\int_0^T\int_{\omega_0}\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt \le \int_0^T\int_{\mathbb{T}^2}\chi\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt. \tag{3.19}$$

Using (1.36), we obtain that

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2}\chi\lambda^4(s\theta)^3|v|^2e^{-2s\varphi}dxdt &= \int_0^T\int_{\mathbb{T}^2}\chi\lambda^4(s\theta)^3vwe^{-2s\varphi}dxdt \\ &\quad+ \int_0^T\int_{\mathbb{T}^2}\chi\lambda^4(s\theta)^3v(-w_t)e^{-2s\varphi}dxdt \\ &\quad+ \int_0^T\int_{\mathbb{T}^2}\chi\lambda^4(s\theta)^3v(c\cdot\nabla w)e^{-2s\varphi}dxdt \\ &=: M_1+M_2+M_3.\end{aligned}$$

Using the Cauchy–Schwarz inequality and (3.16)–(3.17), we have that for every $\varepsilon > 0$

$$|M_1| \le \varepsilon \int_0^T\int_{\mathbb{T}^2} \lambda^4 (s\theta)^3 |v|^2 e^{-2s\varphi} dxdt + \frac{1}{4\varepsilon}\int_0^T\int_\omega \lambda^4 (s\theta)^3 |w|^2 e^{-2s\varphi} dxdt. \tag{3.20}$$

On the other hand, integrating by parts with respect to t in M_2 yields

$$\begin{aligned} M_2 &= \int_0^T\int_{\mathbb{T}^2} \chi\lambda^4 (s\theta)^3 v_t w e^{-2s\varphi} dxdt + \int_0^T\int_{\mathbb{T}^2} \chi\lambda^4 (3s^3\theta^2\theta_t - 2s^4\varphi_t\theta^3) v w e^{-2s\varphi} dxdt \\ &= M_2^1 + M_2^2. \end{aligned}$$

For M_2^1, we notice that for every $\varepsilon > 0$ we have that

$$|M_2^1| \le \varepsilon \int_0^T\int_{\mathbb{T}^2} (s\theta)^{-1} |v_t|^2 e^{-2s\varphi} dxdt + \frac{1}{4\varepsilon}\int_0^T\int_\omega \lambda^8 (s\theta)^7 |w|^2 e^{-2s\varphi} dxdt. \tag{3.21}$$

Since $|\theta_t| + |\varphi_t| \le C\lambda\theta^2$, we infer that for every $\varepsilon > 0$

$$\begin{aligned} |M_2^2| &\le C \int_0^T\int_{\mathbb{T}^2} \chi s^4 (\lambda\theta)^5 |vw| e^{-2s\varphi} dxdt \\ &\le \varepsilon \int_0^T\int_{\mathbb{T}^2} \lambda^4 (s\theta)^3 |v|^2 e^{-2s\varphi} dxdt + \frac{C}{\varepsilon s^2}\int_0^T\int_\omega \lambda^6 (s\theta)^7 |w|^2 e^{-2s\varphi} dxdt. \end{aligned} \tag{3.22}$$

Similarly, integrating by parts with respect to x in M_3 yields

$$\begin{aligned} M_3 = &- \int_0^T\int_{\mathbb{T}^2} (c\cdot\nabla\chi)\lambda^4 (s\theta)^3 v w e^{-2s\varphi} dxdt - \int_0^T\int_{\mathbb{T}^2} \chi\lambda^4 (s\theta)^3 (c\cdot\nabla v) w e^{-2s\varphi} dxdt \\ &+ \int_0^T\int_{\mathbb{T}^2} \chi\lambda^4 \big(3s^3\theta^2 (c\cdot\nabla\theta) - 2s^4\theta^3 (c\cdot\nabla\varphi)\big) v w e^{-2s\varphi} dxdt \\ = &-M_3^1 - M_3^2 + M_3^2. \end{aligned} \tag{3.23}$$

For any $\varepsilon > 0$, we have that

$$|M_3^1| \le \varepsilon \int_0^T\int_{\mathbb{T}^2} \lambda^4 (s\theta)^3 |v|^2 e^{-2s\varphi} dxdt + \frac{C}{\varepsilon}\int_0^T\int_\omega \lambda^4 (s\theta)^3 |w|^2 e^{-2s\varphi} dxdt \tag{3.24}$$

and

$$|M_3^2| \le \varepsilon \int_0^T\int_{\mathbb{T}^2} \lambda^2 (s\theta) |\nabla v|^2 e^{-2s\varphi} dxdt + \frac{C}{\varepsilon}\int_0^T\int_\omega \lambda^6 (s\theta)^5 |w|^2 e^{-2s\varphi} dxdt. \tag{3.25}$$

Finally, since $|\nabla\theta| + |\nabla\varphi| \le C\lambda\theta$, we have that

$$\begin{aligned} |M_3^3| &\le C \int_0^T\int_{\mathbb{T}^2} \chi\lambda^5 (s\theta)^4 |vw| e^{-2s\varphi} dxdt \\ &\le \varepsilon \int_0^T\int_{\mathbb{T}^2} \lambda^4 (s\theta)^3 |v|^2 e^{-2s\varphi} dxdt + \frac{C}{\varepsilon}\int_0^T\int_\omega \lambda^6 (s\theta)^5 |w|^2 e^{-2s\varphi} dxdt. \end{aligned} \tag{3.26}$$

Gathering together (3.15)–(3.26), taking ε small enough, and noticing that $|g'|\varphi/g \le C\theta^2$, we obtain (3.13). It remains to prove the observability inequality (3.14). Pick any $(v_t, w_T) \in [L^2(\mathbb{T}^2)]^2$, and denote by (v, w) the solution of (1.35)–(1.38). Note that $v \in C([0,T], L^2(\mathbb{T}^2)) \cap L^2(0,T,H^1(\mathbb{T}^2))$ and that $w \in C([0,T], L^2(\mathbb{T}^2))$. Using classical semigroup estimates, one derives at once (3.14) from (3.13). ∎

4. Proofs of Theorems 1.1, 1.2, and 1.3.

4.1. Proof of Theorem 1.1. Pick some open sets ω_1 and ω as in Theorem 1.1, and pick $T > 1 - \min_{[0,1]} \rho_2 + \max_{[0,1]} \rho_1$. Let $\hat{\delta} > 0$ and $\sigma > 0$ be given (small) numbers, to be chosen later on. Set $K := 1 - \hat{\delta}$ and $\tau := 1 + K\sigma - \min_{[0,1]} \rho_2$. Pick a function ψ_0 as given in Proposition 2.1. Let ψ be as in (3.1) with $t_0 := T$ and $\tilde{K}$ large enough so that (3.2) holds. Then for $x \in \mathbb{T}^2 \setminus \omega_0$, we have that

$$-2K(t-T) - c \cdot \nabla\psi_0(x) \geq 2K(T-\sigma) - 2\Big(\tau + \max_{x_2 \in [0,1]} \rho_1(x_2)\Big) \text{ for } t \in (0,\sigma),$$

$$-2K(t-T) - c \cdot \nabla\psi_0(x) \leq 2K\sigma - 2\Big(\tau - 1 + \min_{x_2 \in [0,1]} \rho_2(x_2)\Big) \text{ for } t \in (T-\sigma, T),$$

$$c \cdot \nabla(c \cdot \nabla\psi_0)(x) - 2K = 2 - 2(1-\hat{\delta}) > \hat{\delta}.$$

Then the conditions (3.8)–(3.9) are satisfied, and (3.7) holds provided that

$$T \geq 2\sigma + K^{-1}\Big(1 - \min_{x_2 \in [0,1]} \rho_2(x_2) + \max_{x_2 \in [0,1]} \rho_1(x_2)\Big),$$

a condition which is fulfilled for $\hat{\delta}$ and σ small enough. Thus we can apply Propositions 3.1, 3.2, and 3.3. The observability inequality (3.14) gives the desired null controllability of system (1.31)–(1.34).

4.2. Proof of Theorem 1.2. Let $c = (1, p/q)$, and let ω_0 and ω be as in Theorem 1.2. Let ψ_0 be the function given by Proposition 2.3. Pick any $T > q$. Then we can find some numbers $\tau \in (0,1]$, $\sigma \in (0, \min(1, T/2))$, and $\hat{\delta} \in (0,1)$ such that, taking $K := 1 - \hat{\delta}$, we have

$$T \geq \sigma + \frac{q+\tau}{K}, \tag{4.1}$$

$$\tau - K\sigma \geq 0. \tag{4.2}$$

Let ψ be as in (3.1) with $t_0 := T$ and $\tilde{K}$ large enough so that (3.2) holds. Then for $x \in \mathbb{T}^2 \setminus \omega_0$, we have that

$$-2K(t-T) - c \cdot \nabla\psi_0(x) \geq 2K(T-\sigma) - (2q + 2\tau) \geq 0 \text{ for } t \in (0,\sigma),$$

$$-2K(t-T) - c \cdot \nabla\psi_0(x) \leq 2K\sigma - 2\tau \leq 0 \text{ for } t \in (T-\sigma, T),$$

$$c \cdot \nabla(c \cdot \nabla\psi_0)(x) - 2K = 2 - 2(1-\hat{\delta}) > \hat{\delta}.$$

The conditions (3.7), (3.8), and (3.9) are satisfied. Thus we can apply Propositions 3.1, 3.2, and 3.3. The observability inequality (3.14) gives the desired null controllability of system (1.31)–(1.34).

4.3. Proof of Theorem 1.3. Let $c = (1, \zeta)$ with $\zeta \in \mathfrak{I}_\delta$ and $f_2(\delta, \zeta) < 1$. Pick any nonempty open set ω in $\mathbb{T}^2$. We can assume that $0 \in \omega$ without loss of generality. Pick any open set ω_0 with $0 \in \omega_0$ and $\overline{\omega_0} \subset \omega$. Let the pair (p, q) and the function ψ_0 be as given by Proposition 2.4. Note that

$$|c \cdot \nabla(c \cdot \nabla\psi_0)(x) - 2| \leq f_2(\delta, \zeta) < 1, \quad x \in \mathbb{T}^2 \setminus \omega_0,$$

and hence there is some number $\hat{\delta} \in (0,1)$ such that

$$c \cdot \nabla(c \cdot \nabla\psi_0)(x) > 2 - \hat{\delta}, \quad x \in \mathbb{T}^2 \setminus \omega_0.$$

Pick any $\tau \in (0, 1]$, $\sigma \in (0, \min(1, T/2))$ and $K := 1 - \hat{\delta}$. Let ψ be as in (3.1) with $t_0 := T/2$ and $\tilde{K}$ large enough so that (3.2) holds. Then for $x \in \mathbb{T}^2 \setminus \omega_0$, we have that

$$-2K\Big(t - \frac{T}{2}\Big) - c \cdot \nabla\psi_0(x) \geq 2K\Big(\frac{T}{2} - \sigma\Big) - (2q + 2\tau) \text{ for } t \in (0, \sigma),$$

$$-2K\Big(t - \frac{T}{2}\Big) - c \cdot \nabla\psi_0(x) \leq -2K\Big(\frac{T}{2} - \sigma\Big) - 2\tau \text{ for } t \in (T - \sigma, T),$$

$$c \cdot \nabla(c \cdot \nabla\psi_0)(x) - 2K > 2 - \hat{\delta} - 2(1 - \hat{\delta}) = \hat{\delta}.$$

It is clear that the conditions (3.7), (3.8), and (3.9) are satisfied for $T > 0$ large enough. Thus we can apply Propositions 3.1, 3.2, and 3.3. The observability inequality (3.14) gives the desired null controllability of system (1.31)–(1.34).

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