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Boundary stabilization of a microbeam model

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In this paper, we study the boundary stabilization of the deflection of a clamped-free microbeam, which is modeled by a sixth-order hyperbolic equation. We design a boundary feedback control, simpler than the one designed in Vatankhah et al,[2] that forces the energy associated to the deflection to decay exponentially to zero as the time goes to infinity. The rate in which the energy exponentially decays is explicitly given.

KEYWORDS

boundary stabilization, exponential energy decay, hyperbolic equation, Lyapunov techniques, microbeam model

MSC CLASSIFICATION

35B40; 35L35; 74K10; 93B52

1 | INTRODUCTION

A microbeam is a beam whose structural dimensions are in the order of microns. According to Younis,[1] the microbeams are perhaps the most common structural components used in microelectromechanical systems (MEMS) such as actuators, filters, resonators, and sensors. The study of its control properties is recent. Indeed, the boundary stabilization was addressed in Vatankhah et al,[2,3] the internal stabilization was analyzed in Guzmán,[4] the exact boundary controllability was solved in Guzmán and Zhu,[5]Vatankhah et al,[6] and Zhang and Wang,[7] and the exact boundary observability was studied in Edalatzadeh et al.[8] The purpose of this paper is to improve the boundary stabilization result in Vatankhah et al.[2]

The deflection $z = z(t, x)$ of a clamped-free microbeam of density $\rho > 0$, cross–sectional area $A > 0$, Young modulus $E > 0$, area moment of inertia $I > 0$, shear modulus $G > 0$, and length $L > 0$ can be modeled by the sixth-order hyperbolic equation:

$$\begin{cases} \rho A z_{tt} + M_1 z_{xxxx} - M_2 z_{xxxxxx} = 0, \ (t,x) \in (0,\infty) \times (0,L), \\ z(t,0) = z_x(t,0) = z_{xx}(t,0) = 0, \ t \in (0,\infty), \\ M_1 z_{xxx}(t,L) - M_2 z_{xxxxx}(t,L) = -u(t), \ t \in (0,\infty), \\ M_1 z_{xx}(t,L) - M_2 z_{xxxx}(t,L) = 0, \ t \in (0,\infty), \\ M_2 z_{xxx}(t,L) = 0, \ t \in (0,\infty), \\ z(0,x) = z_0(x), \ x \in (0,L), \\ z_t(0,x) = z_1(x), \ x \in (0,L). \end{cases} \tag{1.1}$$

At the free end of the microbeam, it is assumed that a boundary shear load $u = u(t)$ is acting. The model described by Equation (1.1) has been derived in Kahrobaiyan et al[9] and Kong et al[10]by means of the modified strain gradient theory for linear elasticity developed in Lam et al[11] and Hamilton's Principle. The remaining constants in Equation (1.1) are given by

$$M_1 = EI + GA(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2), \quad M_2 = GA(2l_0^2 + \frac{4}{5}l_1^2).$$

The nonnegative constants (l_0, l_1, l_2) were introduced by Lam et al[11] to characterize the phenomenon observed in experiments that the deformation of some materials is size-dependent. When the structural size of a microbeam is no longer in the order of microns, then those parameters may be considered zero. Further information on microbeams, its related experiments, and also on MEMS may be consulted in other studies[2,9-11] and the references therein.

In order to ease the notation, we omit the writing of $t \in (0,\infty)$ and $x \in (0,L)$. For a regular enough solution $z = z(t,x)$ of Equation (1.1), we define its energy by

$$E(t) = \frac{1}{2}\int_0^L \left(\rho A|z_t|^2 + M_1|z_{xx}|^2 + M_2|z_{xxx}|^2\right) dx. \tag{1.2}$$

Let us consider the following formal computations. Taking the derivative in Equation (1.2) and then performing three integrations by parts we arrive at

$$\begin{aligned} E'(t) &= -\left(M_1 z_{xxx} - M_2 z_{xxxxx}\right) z_t|_{x=0}^{x=L} + \left(M_1 z_{xx} - M_2 z_{xxxx}\right) z_{xt}|_{x=0}^{x=L} + M_2 z_{xxx} z_{xxt}|_{x=0}^{x=L} \\ &= u(t) z_t(t,L), \end{aligned} \tag{1.3}$$

from which it follows that the energy (1.2) is conserved when $u = u(t)$ is not acting, that is to say, when it is zero. With the aim to stabilize $z = z(t,x)$ to the rest position, in Vatankhah et al,[2, Theorem 1] it is obtained that the exponential decay of the energy with the boundary feedback control

$$u(t) = -G_1 z_x(t,L) - G_2 z_t(t,L) \quad \text{with} \quad (G_1,G_2) \in \mathbb{R}^2 \quad \text{such that.1, (26) and (36)} \tag{1.4}$$

Claim 1.1. Let us consider Equation (1.4). If $G_1 = 0$, then it is not allowed $G_2 = 0$ and it does not exist $G_2 \neq 0$ such that.[2, (26) and (36)]

Proof. If $G_1 = 0$, then from Vatankhah et al,[2, (26) and (36)] we see that we need to find $\alpha > 0$ and $G_2 < 0$ such that

$$\alpha < \frac{1}{\max\left\{\frac{16L}{4M_1+M_2}, \frac{L}{\rho A}\right\}} = \alpha_M \qquad \text{and} - G_2 + \frac{\alpha L}{2} + G_2\frac{\alpha L}{2\rho A} \le 0.$$

Let us analyze $\frac{\alpha L}{2} \le G_2\left(1 - \frac{\alpha L}{2\rho A}\right)$ for different values of $\alpha \in (0,\alpha_M)$. If $\alpha \in (0,\alpha_M)$ satisfies $1 - \frac{\alpha L}{2\rho A} > 0$, which is possible by choosing it small enough, then we would conclude that $G_2 > 0$, thus contradicting the requirement $G_2 < 0$. We conclude the claim since it does not exist $\alpha \in (0,\alpha_M)$ such that $1 - \frac{\alpha L}{2\rho A} < 0$. □

Let us note that Claim 1.1 tells us that the boundary feedback control designed in Vatankhah et al,[2, Theorem 1] which is the one given by Equation (1.4), only works when $G_1 \neq 0$. The purpose of this paper is to show that we still can achieve the exponential decay of the energy with the boundary feedback control

$$u(t) = -G_0 z_t(t,L) \quad \text{for any} \quad G_0 > 0. \tag{1.5}$$

In order to present that result we need to introduce

$$\mathcal{V} = \left\{v \in H^3(0,L) \quad / \quad v(0) = 0, \quad v'(0) = 0, \quad v''(0) = 0\right\}, \tag{1.6}$$

which is a Hilbert space equipped with the usual inner product of $H^3(0,L)$. The main result of this paper is the following one.

Theorem 1.1. *For $G_0 > 0$ let us set*

$$\varepsilon_0 = \frac{1}{2\max\left\{L, \frac{L^3\rho A}{2M_1}\right\}}, \quad \varepsilon_1 = \frac{G_0}{\frac{L^3}{4M_1}G_0^2 + \frac{\rho AL}{2}}. \tag{1.7}$$

Let $(z_0, z_1) \in \mathcal{V} \times L^2(0, L)$. *Then, the unique weak solution* $z \in C([0,\infty); \mathcal{V}) \cap C^1([0,\infty); L^2(0,L))$ *of Equation (1.1) with the boundary feedback control (1.5) satisfies*

$$E(t) \leq 3 \exp\left\{-\frac{2}{3}\varepsilon t\right\} E(0), \quad \varepsilon = \min\{\varepsilon_0, \varepsilon_1\}. \tag{1.8}$$

This paper is organized as follows.

- The well-posedness of Equation (1.1) with the boundary feedback control (1.5) is shown in Section 2.
- The proof of Theorem 1.1 is presented in Section 3.

Remark 1. The proof we give for Theorem 1.1 is based on Lyapunov techniques. Another proof might be given by following the Riesz basis approach. The reader may consult[12] for the details of that approach.

2 | WELL-POSEDNESS

In this section, we apply the semigroup theory to prove the well-posedness of Equation (1.1) with the boundary feedback control (1.5). We begin with a lemma that gathers some known inequalities that will be useful for our purposes.

Lemma 2.1. *Let* $f \in H^1(0, L)$ *be such that* $f(0) = 0$. *Then, we have the trace and Poincaré inequalities given respectively by*

$$|f(L)|^2 \leq L\|f'\|^2_{L^2(0,L)} \quad \text{and} \quad \|f\|^2_{L^2(0,L)} \leq \frac{L^2}{2}\|f'\|^2_{L^2(0,L)}. \tag{2.1}$$

Let us introduce some notation. Let us define the Hilbert space $\mathcal{H} = \mathcal{V} \times L^2(0, L)$ equipped with the inner product

$$((z, y), (Z, Y))_{\mathcal{H}} = \int_0^L \left(M_1 z'' Z'' + M_2 z''' Z'''\right) dx + \int_0^L \rho A y Y \, dx,$$

which is well-defined as inner product due to the Poincaré inequality in Equation (2.1). Finally, for $G_0 > 0$, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\begin{aligned}
\mathcal{A}(z, y) &= \left(y, -\frac{M_1}{\rho A} z'''' + \frac{M_2}{\rho A} z''''''\right), \\
D(\mathcal{A}) &= \Big\{(z, y) \in \mathcal{V} \times \mathcal{V} \quad / \quad M_1 z'''' - M_2 z'''''' \in L^2(0, L), \\
&\quad M_1 z'''(L) - M_2 z'''''(L) = G_0 y(L), \quad M_1 z''(L) - M_2 z''''(L) = 0, \quad M_2 z'''(L) = 0\Big\}.
\end{aligned}$$

Remark 2. $D(\mathcal{A}) \subset (H^6(0, L) \cap \mathcal{V}) \times \mathcal{V}$. Indeed, if $z \in \mathcal{V}$ is such that $M_1 z'''' - M_2 z'''''' \in L^2(0, L)$, then we can deduce that $z \in H^6(0, L) \cap \mathcal{V}$ (see Brezis[13, Chapter 8, Remark 7] for instance).

With the previous notations in mind, we see that Equation (1.1) with the boundary feedback control (1.5) can be put into the form $\frac{d}{dt}(z, z_t) - \mathcal{A}(z, z_t) = (0, 0)$ with $(z, z_t)(0) = (z_0, z_1)$. We have the following result.

Proposition 2.1. *(a)* $D(\mathcal{A})$ *is dense in* $\mathcal{H}$. *(b)* $\mathcal{A}$ *generates a contraction semigroup in* $\mathcal{H}$.

Proof. Let us prove that the operator $\mathcal{A}$ is m–dissipative. Let $(z, y) \in D(\mathcal{A})$. After three integrations by parts, we have

$$\begin{aligned}
(\mathcal{A}(z, y), (z, y))_{\mathcal{H}} &= \int_0^L \left(M_1 y'' z'' + M_2 y''' z'''\right) dx + \int_0^L \rho A \left(-\frac{M_1}{\rho A} z'''' + \frac{M_2}{\rho A} z''''''\right) y \, dx \\
&= -\left(M_1 z''' - M_2 z'''''\right) y\Big|_{x=0}^{x=L} + \left(M_1 z'' - M_2 z''''\right) y'\Big|_{x=0}^{x=L} + M_2 z''' y''\Big|_{x=0}^{x=L} \\
&= -G_0 |y(L)|^2 \leq 0,
\end{aligned} \tag{2.2}$$

from which it follows that the operator $\mathcal{A}$ is dissipative in virtue of Cazenave and Haraux.[14, Proposition 2.4.2] Let us note that each term in Equation (2.2) is well-defined because $D(\mathcal{A}) \subset (H^6(0, L) \cap \mathcal{V}) \times \mathcal{V}$ and $H^n(0, L) \subset C^{n-1}([0, L])$ for each $n \in \mathbb{N}$. In order to prove that the operator $\mathcal{A}$ is m–dissipative, by Cazenave and Haraux,[14, Proposition 2.2.6] we need to prove that there exists $\lambda_0 > 0$ such that for every $(z_0, y_0) \in \mathcal{H}$, there exists $(z, y) \in D(\mathcal{A})$ solving

$$(z, y) - \lambda_0 \mathcal{A}(z, y) = (z_0, y_0)\,. \tag{2.3}$$

To do the latter, we are going to solve a suitable variational problem. For a $(z_0, y_0) \in \mathcal{H}$ given, let us introduce the bilinear and linear forms $b : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and $l : \mathcal{V} \to \mathbb{R}$ respectively by

$$b(u, v) = \int_0^L \frac{M_1}{\rho A} u'' v''\, dx + \int_0^L \frac{M_2}{\rho A} u''' v'''\, dx + \int_0^L uv\, dx + \frac{1}{\rho A} G_0 u(L) v(L), \tag{2.4}$$

$$l(v) = \int_0^L (z_0 + y_0)\, v\, dx + \frac{1}{\rho A} G_0 z_0(L) v(L). \tag{2.5}$$

On the one hand, the continuity of the bilinear and linear forms is due to the trace inequality in Equation (2.1) and the Cauchy-Schwarz inequality. On the other hand, there exists $C > 0$ such that $b(v, v) \geq C \|v\|^2_{\mathcal{V}}$ for any $v \in \mathcal{V}$ due to the Poincaré inequality in Equation (2.1). Therefore, the Lax-Milgram Theorem gives us the existence of a unique $u \in \mathcal{V}$ satisfying

$$b(u, v) = l(v), \quad \forall v \in \mathcal{V}. \tag{2.6}$$

After one integration by parts in Equation (2.6), we get

$$\int_0^L \left(-\frac{M_1}{\rho A} u' + \frac{M_2}{\rho A} u''' \right) v'''\, dx = (-1)^3 \int_0^L (-z_0 - y_0 + u) v\, dx, \quad \forall v \in C_0^\infty(0, L),$$

and since $(-z_0 - y_0 + u) \in L^2(0, L)$, we infer that $\left(-M_1 u' + M_2 u'''\right)''' = \rho A (z_0 + y_0 - u)$ in the sense of distributions, allowing us to deduce that $u \in H^6(0, L) \cap V$ (see Brezis[13, Chapter 8, Remark 7] for instance). Furthermore, we have

$$M_1 u'''' - M_2 u'''''' = \rho A (z_0 + y_0 - u) \quad \text{for almost every} \quad x \in (0, L). \tag{2.7}$$

Taking into account that $H^6(0, L) \subset C^5([0, L])$, performing three integrations by parts in Equation (2.6) and then using Equation (2.7), we arrive at

$$\begin{aligned} &\frac{M_2}{\rho A} u'''(L) v''(L) + \frac{1}{\rho A} \left[M_1 u''(L) - M_2 u''''(L) \right] v'(L) \\ &\quad - \frac{1}{\rho A} \left[M_1 u'''(L) - M_2 u'''''(L) - G_0 u(L) + G_0 z_0(L) \right] v(L) = 0, \quad \forall v \in \mathcal{V}, \end{aligned} \tag{2.8}$$

from which we infer

$$\left. \begin{array}{l} M_1 u'''(L) - M_2 u'''''(L) = G_0 u(L) - G_0 z_0(L), \\ M_1 u''(L) - M_2 u''''(L) = 0, \\ M_2 u'''(L) = 0. \end{array} \right\} \tag{2.9}$$

Let us introduce the elements of $\mathcal{V}$ given by $z = u$ and $y = u - z_0$. Then, Equations (2.7) and (2.9) allow us to deduce that $(z, y) \in D(\mathcal{A})$ solves Equation (2.3) with $\lambda_0 = 1$. Finally, we get that (a) and (b) are consequences of Cazenave and Haraux[14, Corollary 2.4.3, Theorem 3.4.4], respectively. □

3 | BOUNDARY STABILIZATION

In this section, we prove our main result. Following the ideas introduced in Komornik[15] and Komornik and Zuazua,[16] for a regular enough solution $z = z(t, x)$ of Equation (1.1), we define its perturbed energy by

$$P(t) = E(t) + \varepsilon \left(\rho A \int_0^L x z_x z_t\, dx \right), \tag{3.1}$$

where $\varepsilon > 0$ will be chosen later. The next lemma gives a lower and a upper bound for the perturbed energy (3.1) in terms of the energy (1.2).

Lemma 3.1. *Let us assume Equation (1.7). Then, for any $\varepsilon \in (0, \varepsilon_0]$, we have*

$$\frac{1}{2}E(t) \le P(t) \le \frac{3}{2}E(t). \tag{3.2}$$

Proof. The Cauchy inequality and the Poincaré inequality in Equation (2.1) yield

$$\left|\varepsilon\left(\rho A\int_0^L xz_xz_t\,dx\right)\right| \le \varepsilon\max\left\{L, \tfrac{L^3\rho A}{2M_1}\right\}E(t). \tag{3.3}$$

Then, in the perturbed energy (3.1) we apply Equation (3.3) to obtain

$$\left(1-\varepsilon\max\left\{L, \frac{L^3\rho A}{2M_1}\right\}\right)E(t) \le P(t) \le \left(1+\varepsilon\max\left\{L, \frac{L^3\rho A}{2M_1}\right\}\right)E(t), \tag{3.4}$$

from which it follows Equation (3.2) after the choice of Equation (1.7). □

Now, we have all the required elements to give a proof of our main result.

Proof. of Theorem 1.1. For the moment, let us assume that $(z_0, z_1) \in D(\mathcal{A})$, so that Equation (1.1) with the boundary feedback control (1.5) would have a unique strong solution $z = z(t, x)$ satisfying $(z, z_t) \in C([0, \infty); D(\mathcal{A})) \cap C^1([0, \infty); \mathcal{H})$ by (b) of Proposition 2.1 and Cazenave and Haraux[14], Proposition 4.1.6.

Taking the derivative in the perturbed energy (3.1) and then performing one integration by parts, we get

$$\begin{aligned} P'(t) &= E'(t) + \varepsilon\rho A\int_0^L (xz_{xt}z_t + xz_xz_{tt})\,dx \\ &= E'(t) - \frac{\varepsilon}{2}\int_0^L \rho A|z_t|^2dx + \frac{\varepsilon}{2}\,\rho Ax|z_t|^2\Big|_{x=0}^{x=L} + \varepsilon\int_0^L xz_x\left(-M_1z_{xxxx} + M_2z_{xxxxxx}\right)dx. \end{aligned} \tag{3.5}$$

Further integrations by parts yield

$$\begin{aligned} P'(t) &= E'(t) - \varepsilon E(t) - \varepsilon\int_0^L M_1|z_{xx}|^2dx - 2\varepsilon\int_0^L M_2|z_{xxx}|^2dx + \frac{\varepsilon}{2}\,\rho Ax|z_t|^2\Big|_{x=0}^{x=L} \\ &\quad - \varepsilon xz_x\left(M_1z_{xxx} - M_2z_{xxxxx}\right)\Big|_{x=0}^{x=L} + \varepsilon\left(z_x + xz_{xx}\right)\left(M_1z_{xx} - M_2z_{xxxx}\right)\Big|_{x=0}^{x=L} \\ &\quad - \frac{\varepsilon}{2}\,M_1x|z_{xx}|^2\Big|_{x=0}^{x=L} + 2\varepsilon M_2z_{xx}z_{xxx}\Big|_{x=0}^{x=L} + \frac{\varepsilon}{2}\,M_2x|z_{xxx}|^2\Big|_{x=0}^{x=L}. \end{aligned} \tag{3.6}$$

Since the trace inequality in Equation (2.1) implies

$$\frac{M_1}{L}|z_x(t, L)|^2 \le \int_0^L M_1|z_{xx}|^2dx, \tag{3.7}$$

we see that plugging Equation (1.3) into Equation (3.6) and then applying the Cauchy inequality, we get

$$\begin{aligned} P'(t) &\le -\varepsilon E(t) - \varepsilon\frac{M_1}{L}|z_x(t, L)|^2 + \varepsilon\left(-\frac{G_0}{\varepsilon} + \frac{\rho AL}{2}\right)|z_t(t, L)|^2 - \varepsilon Lz_x(t, L)G_0z_t(t, L) \\ &\le -\varepsilon E(t) + \varepsilon\left(\frac{L^3}{4M_1}G_0^2 - \frac{1}{\varepsilon}G_0 + \frac{\rho AL}{2}\right)|z_t(t, L)|^2. \end{aligned} \tag{3.8}$$

In Equation (3.8), we would like to choose $\varepsilon > 0$ so that

$$\frac{L^3}{4M_1}G_0^2 - \frac{1}{\varepsilon}G_0 + \frac{\rho AL}{2} \le 0. \tag{3.9}$$

Therefore, with the choice of Equation (1.7) and $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, we arrive at

$$P'(t) + \varepsilon E(t) \leq 0. \tag{3.10}$$

In Equation (3.10), we apply Equation (3.2) to deduce Equation (1.8), which is the exponential decay of the energy. Finally, since $D(\mathcal{A})$ is dense in $\mathcal{H}$ by (a) of Proposition 2.1, we can employ Equation (1.8) together with a density argument to conclude that Equation (1.1) has a unique weak solution $z \in C([0,\infty); \mathcal{V}) \cap C^1([0,\infty); L^2(0,L))$, satisfying Equation (1.8), provided that $(z_0, z_1) \in \mathcal{H}$. □

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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