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Local Exact Controllability to the Trajectories of the Cahn–Hilliard Equation

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Abstract In this paper we prove the local exact controllability to the trajectories of the Cahn–Hilliard equation, which is a nonlinear fourth-order parabolic equation, by means of a control supported on an interior open interval. To prove this result we derive a Carleman estimate that allows us to conclude, thanks to a duality argument, the null controllability of the linearized equation around a given solution. Then, we apply a local inversion theorem to extend the control result to the nonlinear equation.

Keywords Cahn–Hilliard equation · Parabolic equation · Internal control · Null controllability · Carleman estimates

Mathematics Subject Classification 35K35 · 93B05 · 93B07

1 Introduction

Let $T > 0$ and $L > 0$. In this paper we consider the Cahn–Hilliard equation

$$z_t + z_{xxxx} = (\varphi(z))_{xx}\,, \quad (t, x) \in (0, T) \times (0, L). \tag{1.1}$$

This equation has been introduced in [3,4] as a continuum model to describe the phase separation that occurs when an isotropic binary system of nonuniform concentration, with homogeneous free energy $\varphi(z) = z^3 - z$, is cooled sufficiently. Such systems may be binary alloys or binary solutions, for instance. Here $z = z(t, x)$ denotes the

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concentration of one of the two components of the system. In the derivation of (1.1) it is assumed at the boundary of the system that the effects produced by the cooling are negligible [4, p. 259] and that no flux of mass occurs [5, p. 160]. Then, these assumptions are translated into the following boundary conditions.

$$z_x(t,0) = z_x(t,L) = 0, \quad z_{xxx}(t,0) = z_{xxx}(t,L) = 0, \quad t \in (0,T). \tag{1.2}$$

To conclude this brief presentation, we mention that details of this model may be consulted in [3–5,16,27,28] and the references therein.

Let us consider a non-empty open interval $\omega \subset (0,L)$ and let us denote by $\mathbb{1}_\omega$ its characteristic function, that is to say, $\mathbb{1}_\omega(x) = 1$ if $x \in \omega$ and $\mathbb{1}_\omega(x) = 0$ if $x \notin \omega$. In this paper we address the exact controllability to the trajectories of

$$\begin{cases} z_t + z_{xxxx} + z_{xx} = \left(z^3\right)_{xx} + u\mathbb{1}_\omega, & (t,x) \in (0,T)\times(0,L), \\ z_x(t,0) = 0, \ z_x(t,L) = 0, & t \in (0,T), \\ z_{xxx}(t,0) = 0, \ z_{xxx}(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{1.3}$$

In other words, we wonder if given a solution $\overline{z} = \overline{z}(t,x)$ of

$$\begin{cases} \overline{z}_t + \overline{z}_{xxxx} + \overline{z}_{xx} = \left(\overline{z}^3\right)_{xx}, & (t,x) \in (0,T)\times(0,L), \\ \overline{z}_x(t,0) = 0, \ \overline{z}_x(t,L) = 0, & t \in (0,T), \\ \overline{z}_{xxx}(t,0) = 0, \ \overline{z}_{xxx}(t,L) = 0, & t \in (0,T), \end{cases} \tag{1.4}$$

there exists a control $u = u(t,x)$ such that the corresponding solution $z = z(t,x)$ of (1.3) satisfies that $z(T,\cdot) = \overline{z}(T,\cdot)$. When $\overline{z} = 0$ the exact controllability to the trajectories is called null controllability. We do not ask an arbitrary $\overline{z} = \overline{z}(t,x)$ due to the parabolic character of (1.3).

Our main result establishes the local exact controllability to the trajectories of (1.3), local in the sense that we ask $\|z_0 - \overline{z}(0,\cdot)\|_{L^2(0,L)}$ to be small enough.

Theorem 1.1 *Let $\overline{z} \in L^\infty(0,T;W^{2,\infty}(0,L))$ be a given solution of (1.4). There exists $\varepsilon > 0$ such that for any $z_0 \in L^2(0,L)$ with $\|z_0 - \overline{z}(0,\cdot)\|_{L^2(0,L)} < \varepsilon$ we have that there exists a control $u \in L^2(0,T;L^2(\omega))$ such that the unique solution $z \in C([0,T];L^2(0,L))$ of (1.3) satisfies that $z(T,\cdot) = \overline{z}(T,\cdot)$.*

The analysis required to prove this result is not performed directly to (1.3), but rather to the control system satisfied by $y = z - \overline{z}$, namely

$$\begin{cases} \mathcal{L}y = \mathcal{N}y + u\mathbb{1}_\omega, & (t,x) \in (0,T)\times(0,L), \\ y_x(t,0) = 0, \ y_x(t,L) = 0, & t \in (0,T), \\ y_{xxx}(t,0) = 0, \ y_{xxx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases} \tag{1.5}$$

where

$$\mathcal{L}y = y_t + y_{xxxx} + \left(-3\overline{z}^2 + 1\right) y_{xx} + \left(-12\overline{z}\overline{z}_x\right) y_x + \left(-6\overline{z}\overline{z}_{xx} - 6\overline{z}_x^2\right) y, \tag{1.6}$$

$$\mathcal{N}y = (3\overline{z}_{xx})\, y^2 + (6\overline{z})\, y_x^2 + (12\overline{z}_x)\, yy_x + (6\overline{z})\, yy_{xx} + (6)\, yy_x^2 + (3)\, y^2 y_{xx}, \tag{1.7}$$
$$y_0(\cdot) = z_0(\cdot) - \overline{z}(0,\cdot).$$

We see that the exact controllability to the trajectories of (1.3) is equivalent to the null controllability of (1.5). Accordingly, we focus on the proof of the latter, which can be done by following a two steps strategy developed in [20,26].

- **Step 1** For an adequate choice of $f = f(t,x)$ we prove the null controllability of

$$\begin{cases} \mathcal{L}y = f + u\mathbb{1}_\omega, & (t,x) \in (0,T)\times(0,L), \\ y_x(t,0) = 0,\ \ y_x(t,L) = 0, & t \in (0,T), \\ y_{xxx}(t,0) = 0,\ \ y_{xxx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{1.8}$$

  This is done in Theorem 4.1 by means of a new Carleman estimate and a duality argument. In the context of fourth-order parabolic equations, these estimates have been used for studying problems of internal control [7,12,13,21–23,33], boundary control [9–11] and stability of inverse problems [2,22,25]. However, none of them is suitable for us because of the boundary conditions in (1.8). For this reason in Proposition 3.1 we derive a new Carleman estimate, in which it was necessary to construct a new weight function and to adapt the approach of Fursikov and Imanuvilov for Carleman estimates in [20]. This is one of the novelties of this paper, since the weight functions and methods employed in other papers dealing with fourth-order parabolic equations were not suitable for us.
- **Step 2** Then, we apply a local inversion theorem, which can be found in [1,20], to extend the control result from (1.8) to (1.5). Because of such a local inversion theorem we had to ask the smallness condition for $\|z_0 - \overline{z}(0,\cdot)\|_{L^2(0,L)}$ in Theorem 1.1.

This two steps strategy has been successfully applied to others nonlinear parabolic equations or systems such as the Boussinesq system [6,19,24], the Kuramoto–Sivashinsky equation [7,10–12] and the Navier–Stokes system [8,18,19,26]. We highlight that the Kuramoto–Sivashinsky equation is also a nonlinear fourth-order parabolic equation, but with nonlinearity and boundary conditions different to the ones of the Cahn–Hilliard equation. Even if this two steps strategy is classical, each step has its own difficulties depending on the equation or system under consideration.

Let us make a comment regarding Theorem 1.1. Formal computations in (1.3) yield

$$\frac{d}{dt}\left(\int_0^L z(t,x)\,dx\right) = \int_\omega u(t,x)\,dx, \quad t \in (0,T).$$

Therefore, in view of (1.1) and (1.2) we see that $z = z(t,x)$ is conserved. In Theorem 1.1 we have not been able to construct controls $u = u(t,x)$ preserving such a conservation.

Regarding the study of the control properties of the Cahn–Hilliard equation we mention, without being exhaustive, the following works. The approximate controllability is considered in [15]. The question of the existence of insensitizing controls is answered in [21]. The stabilization problem with feedback controls together with an optimal control problem are addressed in [30]. The existence of time optimal controls is shown in [31]. Finally, the asymptotic behavior of solutions is studied in [32].

Remark 1.1 The boundary conditions considered in [15,21,23,31] are different to the ones in (1.2), that is to say, are different to the ones associated to the Cahn–Hilliard equation.

The rest of this paper is organized as follows. In Sect. 2 we present the well-posedness result needed here. In Sect. 3 we derive the new Carleman estimate. In Sect. 4.1 we prove the null controllability of (1.8) and in Sect. 4.2 we prove the local exact controllability to the trajectories of (1.3). Finally, in Sect. 5 we present further remarks.

2 Well-Posedness

In this short section we just present a well-posedness result for

$$\begin{cases} y_t + y_{xxxx} + a_2 y_{xx} + a_1 y_x + a_0 y = f, & (t,x) \in (0,T) \times (0,L), \\ y_x(t,0) = 0, \;\; y_x(t,L) = 0, & t \in (0,T), \\ y_{xxx}(t,0) = 0, \;\; y_{xxx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{2.1}$$

To this end we need to introduce

$$\mathcal{H}_2 = \{h \in H^2(0,L) \;/\; h'(0) = h'(L) = 0\}, \tag{2.2}$$

which is well defined thanks to the continuous injection $H^2(0,L) \hookrightarrow C^1([0,L])$. Furthermore, $\mathcal{H}_2$ is closed in $H^2(0,L)$, thus it is a Banach space endowed with the usual norm of $H^2(0,L)$.

Applying the methods employed to prove [17, Theorem 2.1] and [29, Chapter 3, Theorem 4.2] for the Cahn–Hilliard equation, we can obtain the following result, whose proof is omitted for being classical.

Proposition 2.1 *Let* $(a_2,a_1,a_0) \in L^\infty(0,T;L^\infty(0,L))^3$, $f \in L^1(0,T;L^2(0,L))$ *and* $y_0 \in L^2(0,L)$. *Then,* (2.1) *has a unique solution* $y \in C([0,T];L^2(0,L)) \cap L^2(0,T;\mathcal{H}_2)$. *Moreover, there exists* $C > 0$ *such that*

$$\|y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))} \le C\|f\|_{L^1(0,T;L^2(0,L))} + C\|y_0\|_{L^2(0,L)}. \tag{2.3}$$

The importance of this result will be made clear in Sect. 4, in particular for the proof of the null controllability of (1.8).

3 Carleman Estimate

In this section we adapt the approach of Fursikov and Imanuvilov in [20] for deriving Carleman estimate in order to derive ours.

3.1 Weight Functions

We proceed to construct suitable weight functions. Let us recall that $\omega \subset (0, L)$ is a non-empty open interval. For a fixed $x_0 \in int(\omega)$ let us introduce $\omega_0 = (x_0 - \varepsilon, x_0 + \varepsilon)$ with $\varepsilon = dist(x_0, \partial\omega)/2 > 0$, thus obtaining that $\overline{\omega_0} \subset int(\omega)$. Then, let us take a $\psi \in C^4([0, L])$ satisfying:

- $\psi > 0 \quad \text{in} \quad (0, L).$ (3.1)
- $\psi(0) = \psi(L) = 0 \quad \text{and} \quad \psi''(0) = \psi''(L) = 0.$ (3.2)
- $|\psi'| > 0 \quad \text{in} \quad [0, L]\backslash\omega_0.$ (3.3)

Remark 3.1 For instance, we can take a non-negative $\psi \in C^4([0, L])$ so that $\psi(x) = x$ when $x \in [0, x_0 - \varepsilon]$ and $\psi(x) = L - x$ when $x \in [x_0 + \varepsilon, L]$.

Finally, for $\mu > 0$ let us define the weight functions

$$\alpha(t, x) = \frac{e^{4\mu\|\psi\|_{L^\infty(0,L)}}}{t(T-t)} - \beta(t, x), \quad \beta(t, x) = \frac{e^{2\mu\|\psi\|_{L^\infty(0,L)} + \mu\psi(x)}}{t(T-t)}, \\ (t, x) \in [0, T] \times [0, L]. \tag{3.4}$$

Some useful properties for these weight functions will be mentioned when proving our Carleman estimate. However, now we mention that they satisfy

$$\lim_{t \to 0^+} \alpha(t, \cdot) = \lim_{t \to T^-} \alpha(t, \cdot) = \lim_{t \to 0^+} \beta(t, \cdot) = \lim_{t \to T^-} \beta(t, \cdot) = \infty. \tag{3.5}$$

3.2 A Fourth-Order Parabolic Operator

Let us introduce the notation $Q = (0, T) \times (0, L)$ and $Q_\omega = (0, T) \times \omega$. Let us consider the operator $\mathcal{L}^* q = -q_t + q_{xxxx} + A_2 q_{xx} + A_1 q_x + A_0 q$, with $A_i \in L^\infty(0, T; L^\infty(0, L))$ for $i \in \{0, 1, 2\}$ and $q = q(t, x)$ belonging to

$$\mathcal{Q} = \{q \in C^\infty([0, T] \times [0, L]) \ / \ q_x(\cdot, 0) = q_x(\cdot, L) = 0 \\ \text{and} \quad q_{xxx}(\cdot, 0) = q_{xxx}(\cdot, L) = 0\}. \tag{3.6}$$

Our Carleman estimate is the following one.

Proposition 3.1 *There exist* $\mu_0 \geq 1$ *and* $C > 0$*, both independent of* $(T, \mu, \nu) \in (0, \infty)^3$*, such that for any* $\mu \geq \mu_0$ *and* $\nu \geq \max\{T, T^2\}$ *we have that* $q \in \mathcal{Q}$ *satisfies*

$$\iint_Q e^{-2\nu\alpha}\left(\nu^7\mu^8\beta^7|q|^2+\nu^5\mu^6\beta^5|q_x|^2+\nu^3\mu^4\beta^3|q_{xx}|^2+\nu\mu^2\beta|q_{xxx}|^2\right)dxdt$$
$$+\iint_Q e^{-2\nu\alpha}\left(\frac{|q_t|^2+|q_{xxxx}|^2}{\nu\beta}\right)dxdt\le C\iint_Q e^{-2\nu\alpha}|\mathcal{L}^*q|^2dxdt$$
$$+C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt. \tag{3.7}$$

Proof We are going to derive two Carleman estimates, with similar weight functions, that later will be added to cancel some unwanted boundary terms that will arise in the computations.

For $\nu>0$ and $w=e^{-\nu\alpha}q$ let us consider $\mathcal{P}w=e^{-\nu\alpha}\mathcal{L}^*\left(e^{\nu\alpha}w\right)$ and the decomposition $\mathcal{P}w=\mathcal{P}_1w+\mathcal{P}_2w+\mathcal{P}_3w$ given by

$$\begin{aligned}
\mathcal{P}_1w=&-w_t+4\nu^3\alpha_x^3w_x+4\nu\alpha_xw_{xxx}+3\nu^3\alpha_x\left(\alpha_x^2\right)_x w,\\
\mathcal{P}_2w=&\nu^4\alpha_x^4w+6\nu^2\alpha_x^2w_{xx}+w_{xxxx}+6\nu^2\left(\alpha_x^2\right)_x w_x,\\
\mathcal{P}_3w=&-\nu\alpha_tw+3\nu^2\alpha_{xx}^2w+4\nu^2\alpha_{xxx}\alpha_xw+\nu\alpha_{xxxx}w+4\nu\alpha_{xxx}w_x+6\nu\alpha_{xx}w_{xx}\\
&+A_2\left(\nu^2\alpha_x^2w+\nu\alpha_{xx}w+2\nu\alpha_xw_x+w_{xx}\right)+A_1\left(\nu\alpha_xw+w_x\right)+A_0w.
\end{aligned}$$

The choice of a suitable decomposition for $\mathcal{P}w$ is a delicate part when deriving Carleman estimates. Regardless of the decomposition that we could choose, its structure always satisfies

$$\|\mathcal{P}_1w\|^2_{L^2(Q)}+2\left(\mathcal{P}_1w,\mathcal{P}_2w\right)_{L^2(Q)}+\|\mathcal{P}_2w\|^2_{L^2(Q)}=\|\mathcal{P}w-\mathcal{P}_3w\|^2_{L^2(Q)}. \tag{3.8}$$

Throughout the proof, which we split it into six parts for a better comprehension, it will appear several constants independent of $(T,\mu,\nu)\in(0,\infty)^3$ that may vary from line to line.

Part 1 In this part we handle the terms $\|\mathcal{P}_1w\|^2_{L^2(Q)}$ and $\|\mathcal{P}_2w\|^2_{L^2(Q)}$. To this end it is convenient to introduce

$$I(w,\beta)=\iint_Q\left(\nu^7\mu^8\beta^7|w|^2+\nu^5\mu^6\beta^5|w_x|^2+\nu^3\mu^4\beta^3|w_{xx}|^2\right.$$
$$\left.+\nu\mu^2\beta|w_{xxx}|^2\right)dxdt, \tag{3.9}$$
$$J(w,\beta)=\iint_Q\left(\frac{|w_t|^2+|w_{xxxx}|^2}{\nu\beta}\right)dxdt. \tag{3.10}$$

Since $\alpha_x=-\mu\psi'\beta$ and $\alpha_{xx}=-\mu\psi''\beta-\mu^2\left(\psi'\right)^2\beta$, we can get the following inequalities.

$$\iint_Q \frac{|w_t|^2}{\nu\beta} dxdt \leq C_1 \iint_Q \frac{|\mathcal{P}_1 w|^2}{\nu\beta} dxdt + C_1 \iint_Q \Big(\nu^5\mu^6\beta^5|w_x|^2 + \nu\mu^2\beta|w_{xxx}|^2 + \nu^5\mu^6\beta^5|w|^2 + \nu^5\mu^8\beta^5|w|^2\Big)\, dxdt,$$

$$\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta} dxdt \leq C_1 \iint_Q \frac{|\mathcal{P}_2 w|^2}{\nu\beta} dxdt + C_1 \iint_Q \Big(\nu^7\mu^8\beta^7|w|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu^3\mu^4\beta^3|w_x|^2 + \nu^3\mu^6\beta^3|w_x|^2\Big)\, dxdt.$$

Adding the previous two inequalities and then employing that $\beta \geq 4/T^2$ it follows

$$J(w,\beta) \leq C_1\|\mathcal{P}_1 w\|^2_{L^2(Q)} + C_1\|\mathcal{P}_2 w\|^2_{L^2(Q)} + C_1 I(w,\beta) \quad \text{for} \quad (\mu,\nu) \in [1,\infty)\times[T^2,\infty). \tag{3.11}$$

Part 2 In this part we compute $(\mathcal{P}_1 w, \mathcal{P}_2 w)_{L^2(Q)}$. For $(i,j) \in \{1,2,3,4\}^2$ let us denote by $I_{i,j}$ the L^2—product in Q between the ith term of $\mathcal{P}_1 w$ with the jth term of $\mathcal{P}_2 w$. Then, with this notation we can write

$$(\mathcal{P}_1 w, \mathcal{P}_2 w)_{L^2(Q)} = \sum_{i,j=1}^{4} I_{i,j}. \tag{3.12}$$

Integrations by parts on Q are performed and some of the resulting terms are labeled with the symbol $^{(*)}$ to indicate that later are going to be included in (3.14). In the computations we use that $w(0,\cdot) = w(T,\cdot) = 0$, which is a consequence of $w = e^{-\nu\alpha}q$ together with (3.5). Each resulting expression for $I_{i,j}$ is listed below.

$$I_{1,1} = \frac{\nu^4}{2}\iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt.$$

$$I_{1,2} = 6\nu^2 \iint_Q \left(\alpha_x^2\right)_x w_t w_x\, dxdt - 3\nu^2 \iint_Q \left(\alpha_x^2\right)_t |w_x|^2 dxdt - 6\nu^2 \int_0^T \alpha_x^2\, w_t w_x|_{x=0}^{x=L}\, dt.$$

$$I_{1,3} = \int_0^T w_{tx}w_{xx}|_{x=0}^{x=L}\, dt - \int_0^T w_t w_{xxx}|_{x=0}^{x=L}\, dt.$$

$$I_{1,4} = -6\nu^2 \iint_Q \left(\alpha_x^2\right)_x w_t w_x\, dxdt.$$

$$I_{2,1} = -2\nu^7 \iint_Q \left(\alpha_x^7\right)_x |w|^2 dxdt\ ^{(*)} + 2\nu^7 \int_0^T \alpha_x^7\, |w|^2\Big|_{x=0}^{x=L} dt.$$

$$I_{2,2} = -12\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 dxdt \ ^{(*)} + 12\nu^5 \int_0^T \alpha_x^5 \ |w_x|^2\Big|_{x=0}^{x=L} dt.$$

$$\begin{aligned} I_{2,3} = &-2\nu^3 \iint_Q \left(\alpha_x^3\right)_{xxx} |w_x|^2 dxdt + 6\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 dxdt \ ^{(*)} \\ &+ 2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2\Big|_{x=0}^{x=L} dt \\ &- 2\nu^3 \int_0^T \alpha_x^3 \ |w_{xx}|^2\Big|_{x=0}^{x=L} dt - 4\nu^3 \int_0^T \left(\alpha_x^3\right)_x \ w_x w_{xx}|_{x=0}^{x=L} dt \\ &+ 4\nu^3 \int_0^T \alpha_x^3 \ w_x w_{xxx}|_{x=0}^{x=L} dt. \end{aligned}$$

$$I_{2,4} = 24\nu^5 \iint_Q \alpha_x^3 \left(\alpha_x^2\right)_x |w_x|^2 dxdt \ ^{(*)}.$$

$$\begin{aligned} I_{3,1} = &-2\nu^5 \iint_Q \left(\alpha_x^5\right)_{xxx} |w|^2 dxdt + 6\nu^5 \iint_Q \left(\alpha_x^5\right)_x |w_x|^2 dxdt \ ^{(*)} \\ &+ 2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2\Big|_{x=0}^{x=L} dt \\ &- 2\nu^5 \int_0^T \alpha_x^5 \ |w_x|^2\Big|_{x=0}^{x=L} dt - 4\nu^5 \int_0^T \left(\alpha_x^5\right)_x \ w w_x|_{x=0}^{x=L} dt \\ &+ 4\nu^5 \int_0^T \alpha_x^5 \ w w_{xx}|_{x=0}^{x=L} dt. \end{aligned}$$

$$I_{3,2} = -12\nu^3 \iint_Q \left(\alpha_x^3\right)_x |w_{xx}|^2 dxdt \ ^{(*)} + 12\nu^3 \int_0^T \alpha_x^3 \ |w_{xx}|^2\Big|_{x=0}^{x=L} dt.$$

$$I_{3,3} = -2\nu \iint_Q \alpha_{xx} |w_{xxx}|^2 dxdt \ ^{(*)} + 2\nu \int_0^T \alpha_x \ |w_{xxx}|^2\Big|_{x=0}^{x=L} dt.$$

$$\begin{aligned} I_{3,4} = &\ 12\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} |w_x|^2 dxdt - 24\nu^3 \iint_Q \alpha_x \left(\alpha_x^2\right)_x |w_{xx}|^2 dxdt \ ^{(*)} \\ &+ 24\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x \ w_x w_{xx}|_{x=0}^{x=L} dt - 12\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x \ |w_x|^2\Big|_{x=0}^{x=L} dt. \end{aligned}$$

$$I_{4,1} = 3\nu^7 \iint_Q \alpha_x^5 \left(\alpha_x^2\right)_x |w|^2 dxdt \ ^{(*)}.$$

$$\begin{aligned} I_{4,2} = &\ 9\nu^5 \iint_Q \left(\alpha_x^3 \left(\alpha_x^2\right)_x\right)_{xx} |w|^2 dxdt - 18\nu^5 \iint_Q \alpha_x^3 \left(\alpha_x^2\right)_x |w_x|^2 dxdt \ ^{(*)} \\ &- 9\nu^5 \int_0^T \left(\alpha_x^3 \left(\alpha_x^2\right)_x\right)_x \ |w|^2\Big|_{x=0}^{x=L} dt + 18\nu^5 \int_0^T \alpha_x^3 \left(\alpha_x^2\right)_x \ w w_x|_{x=0}^{x=L} dt. \end{aligned}$$

$$\begin{aligned} I_{4,3} = &-3\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} |w_x|^2 dxdt + 3\nu^3 \iint_Q \alpha_x \left(\alpha_x^2\right)_x |w_{xx}|^2 dxdt \ ^{(*)} \\ &+ 3\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} w w_{xx} \, dxdt + 3\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x \ |w_x|^2\Big|_{x=0}^{x=L} dt \end{aligned}$$

$$- 3\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x w w_{xx}|_{x=0}^{x=L} \, dt + 3\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x w w_{xxx}|_{x=0}^{x=L} \, dt$$
$$- 3\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x w_x w_{xx}|_{x=0}^{x=L} \, dt.$$
$$I_{4,4} = -9\nu^5 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x^2\right)_x |w|^2 dx dt + 9\nu^5 \int_0^T \alpha_x \left(\alpha_x^2\right)_x^2 |w|^2 \Big|_{x=0}^{x=L} dt.$$

Plugging the previous sixteen terms into (3.12) we arrive at

$$(\mathcal{P}_1 w, \mathcal{P}_2 w)_{L^2(Q)} = M(w) + B(w, \alpha, L) - B(w, \alpha, 0) + R(w), \tag{3.13}$$

in which we have defined the main terms

$$\begin{aligned} M(w) = &- 8\nu^7 \iint_Q \alpha_x^6 \alpha_{xx} |w|^2 dx dt - 18\nu^5 \iint_Q \alpha_x^4 \alpha_{xx} |w_x|^2 dx dt \\ &- 60\nu^3 \iint_Q \alpha_x^2 \alpha_{xx} |w_{xx}|^2 dx dt - 2\nu \iint_Q \alpha_{xx} |w_{xxx}|^2 dx dt, \end{aligned} \tag{3.14}$$

the boundary terms

$$\begin{aligned} B(w, \alpha, x) = &- 6\nu^2 \int_0^T \alpha_x^2 w_t w_x \, dt + \int_0^T w_{tx} w_{xx} \, dt - \int_0^T w_t w_{xxx} \, dt \\ &+ 2\nu^7 \int_0^T \alpha_x^7 |w|^2 dt + 12\nu^5 \int_0^T \alpha_x^5 |w_x|^2 dt \\ &+ 2\nu^3 \int_0^T \left(\alpha_x^3\right)_{xx} |w_x|^2 dt - 2\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 dt \\ &- 4\nu^3 \int_0^T \left(\alpha_x^3\right)_x w_x w_{xx} \, dt + 4\nu^3 \int_0^T \alpha_x^3 w_x w_{xxx} \, dt \\ &+ 2\nu^5 \int_0^T \left(\alpha_x^5\right)_{xx} |w|^2 dt - 2\nu^5 \int_0^T \alpha_x^5 |w_x|^2 dt \\ &- 4\nu^5 \int_0^T \left(\alpha_x^5\right)_x w w_x \, dt + 4\nu^5 \int_0^T \alpha_x^5 w w_{xx} \, dt \\ &+ 12\nu^3 \int_0^T \alpha_x^3 |w_{xx}|^2 dt + 2\nu \int_0^T \alpha_x |w_{xxx}|^2 dt \\ &- 12\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x |w_x|^2 dt + 24\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x w_x w_{xx} \, dt \\ &- 9\nu^5 \int_0^T \left(\alpha_x^3 \left(\alpha_x^2\right)_x\right)_x |w|^2 dt + 18\nu^5 \int_0^T \alpha_x^3 \left(\alpha_x^2\right)_x w w_x \, dt \\ &+ 3\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x |w_x|^2 dt - 3\nu^3 \int_0^T \left(\alpha_x \left(\alpha_x^2\right)_x\right)_x w w_{xx} \, dt \end{aligned}$$

$$+ 3\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x w w_{xxx}\, dt - 3\nu^3 \int_0^T \alpha_x \left(\alpha_x^2\right)_x w_x w_{xx}\, dt$$
$$+ 9\nu^5 \int_0^T \alpha_x \left(\alpha_x^2\right)_x^2 |w|^2 dt, \tag{3.15}$$

and the rest terms

$$\begin{aligned}R(w) =& \frac{\nu^4}{2} \iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt - 3\nu^2 \iint_Q \left(\alpha_x^2\right)_t |w_x|^2 dxdt \\ &- 2\nu^3 \iint_Q \left(\alpha_x^3\right)_{xxx} |w_x|^2 dxdt - 2\nu^5 \iint_Q \left(\alpha_x^5\right)_{xxx} |w|^2 dxdt \\ &+ 12\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} |w_x|^2 dxdt + 9\nu^5 \iint_Q \left(\alpha_x^3 \left(\alpha_x^2\right)_x\right)_{xx} |w|^2 dxdt \\ &- 3\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} |w_x|^2 dxdt + 3\nu^3 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x\right)_{xx} w w_{xx}\, dxdt \\ &- 9\nu^5 \iint_Q \left(\alpha_x \left(\alpha_x^2\right)_x^2\right)_x |w|^2 dxdt.\end{aligned} \tag{3.16}$$

Part 3 In this part we handle the main terms defined in (3.14). Here, and later on, we require to employ

$$\left.\begin{aligned}\alpha_x &= -\mu\psi'\beta, \\ \alpha_{xx} &= -\mu\psi''\beta - \mu^2 \left(\psi'\right)^2 \beta, \\ \alpha_{xxx} &= -\mu\psi'''\beta - 3\mu^2\psi'\psi''\beta - \mu^3 \left(\psi'\right)^3 \beta.\end{aligned}\right\} \tag{3.17}$$

Using (3.17) and then considering (3.3) we infer

$$\begin{aligned}M(w) \geq& C_2 I(w,\beta) - \frac{1}{\mu} I(w,\beta) \\ &- \iint_{Q_{\omega_0}} \left(\nu^7\mu^8\beta^7|w|^2 + \nu^5\mu^6\beta^5|w_x|^2 \right. \\ &\left. + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right) dxdt.\end{aligned} \tag{3.18}$$

Let us handle the last three terms of the right-hand side of (3.18). Recalling that $\overline{\omega_0} \subset int(\omega)$, let us take a non-empty open interval ω_1 such that $\overline{\omega_0} \subset \omega_1 \subseteq \omega$. Then, let us consider a non-negative function $\chi \in C_0^\infty(\omega_1)$ such that $\chi = 1$ in ω_0. Some integration by parts on ω_1 gives us

$$\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2dxdt = \underbrace{\frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(\beta^5\chi\right)_{xx}|w|^2dxdt}_{K_1} \underbrace{-\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi w_{xx}w\,dxdt}_{K_2}. \tag{3.19}$$

For K_1 we use that $|\beta_x| \le C_3\mu\beta$ and $|\beta_{xx}| \le C_3\mu\beta + C_3\mu^2\beta$ to obtain

$$\begin{aligned} K_1 &= \frac{1}{2}\iint_{Q_{\omega_1}} \nu^5\mu^6\left(20\beta^3\beta_x^2\chi + 5\beta^4\beta_{xx}\chi + 10\beta^4\beta_x\chi' + \beta^5\chi''\right)|w|^2dxdt, \\ &\le \frac{C_3}{\nu^2}\left(1+\frac{1}{\mu}+\frac{1}{\mu^2}\right)\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^5|w|^2dxdt. \end{aligned}$$

Here taking into account that $\mu \le C_3T^2\beta$ we arrive at

$$K_1 \le \frac{C_3}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2dxdt \quad \text{for} \quad (\mu,\nu) \in [1,\infty)\times[T^2,\infty). \tag{3.20}$$

Regarding K_2 we see that applying Cauchy's inequality leads us to

$$\begin{aligned} |K_2| \le{}& \frac{C_3}{\rho_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2dxdt \\ &+ C_3\rho_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt \quad \text{for} \quad \rho_1 \in (0,\infty). \end{aligned} \tag{3.21}$$

Accordingly, combining (3.19)–(3.21) we arrive at

$$\begin{aligned} \iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2dxdt \le{}& \frac{C_3}{\mu^2}\iint_Q \nu^7\mu^8\beta^7|w|^2dxdt \\ &+ \frac{C_3}{\rho_1}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2dxdt \\ &+ C_3\rho_1\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt \\ &\text{for} \quad (\rho_1,\mu,\nu) \in (0,\infty)\times[1,\infty)\times[T^2,\infty). \end{aligned} \tag{3.22}$$

Let us note that the two remaining terms of the right-hand side of (3.18) can be treated as we just did. Then, we can get the following inequalities.

$$\begin{aligned}\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2dxdt \le{}& \frac{C_3}{\mu^2}\iint_Q \nu^5\mu^6\beta^5|w_x|^2dxdt\\ &+\frac{C_3}{\rho_2}\iint_Q \nu\mu^2\beta|w_{xxx}|^2dxdt\\ &+C_3\rho_2\iint_{Q_{\omega_1}} \nu^5\mu^6\beta^5\chi|w_x|^2dxdt\\ &\text{for } (\rho_2,\mu,\nu)\in(0,\infty)\times[1,\infty)\times[T^2,\infty), \end{aligned}\tag{3.23}$$

$$\begin{aligned}\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2dxdt \le{}& \frac{C_3}{\mu^2}\iint_Q \nu^3\mu^4\beta^3|w_{xx}|^2dxdt\\ &+\frac{C_3}{\rho_3}\iint_Q \frac{|w_{xxxx}|^2}{\nu\beta}dxdt\\ &+C_3\rho_3\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2dxdt\\ &\text{for } (\rho_3,\mu,\nu)\in(0,\infty)\times[1,\infty)\times[T^2,\infty). \end{aligned}\tag{3.24}$$

Plugging (3.22) into (3.23) give us

$$\begin{aligned}&\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2dxdt \le \frac{C_3}{\mu^2}(1+\rho_2)I(w,\beta)\\ &+C_3\left(\frac{1}{\rho_2}+\frac{\rho_2}{\rho_1}\right)I(w,\beta)+C_3\rho_1\rho_2\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt\\ &\text{for } (\rho_1,\rho_2,\mu,\nu)\in(0,\infty)\times(0,\infty)\times[1,\infty)\times[T^2,\infty). \end{aligned}\tag{3.25}$$

Then, plugging (3.25) into (3.24) we get

$$\begin{aligned}\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2dxdt \le{}& \frac{C_3}{\mu^2}(1+\rho_3+\rho_2\rho_3)I(w,\beta)\\ &+C_3\left(\frac{\rho_3}{\rho_2}+\frac{\rho_2\rho_3}{\rho_1}\right)I(w,\beta)+\frac{C_3}{\rho_3}J(w,\beta)\\ &+C_3\rho_1\rho_2\rho_3\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt\\ \text{for } (\rho_1,\rho_2,\rho_3,\mu,\nu)\in{}&(0,\infty)\times(0,\infty)\times(0,\infty)\times[1,\infty)\times[T^2,\infty). \end{aligned}\tag{3.26}$$

On the one hand, in (3.25) we see that the choice of $\rho_2=\rho>0$ and $\rho_1=\rho^2$ yield

$$\begin{aligned}\iint_{Q_{\omega_1}} \nu^3\mu^4\beta^3\chi|w_{xx}|^2dxdt \le{}& \frac{C_3}{\mu^2}(1+\rho)I(w,\beta)+\frac{C_3}{\rho}I(w,\beta)\\ &+C_3\rho^3\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt\\ &\text{for } (\rho,\mu,\nu)\in(0,\infty)\times[1,\infty)\times[T^2,\infty). \end{aligned}$$

On the other hand, in (3.26) we see that the choice of $\rho_3 = \rho > 0$, $\rho_2 = \rho^2$ and $\rho_1 = \rho^4$ lead us to

$$\iint_{Q_{\omega_1}} \nu\mu^2\beta\chi|w_{xxx}|^2dxdt \leq \frac{C_3}{\mu^2}(1+\rho+\rho^3)I(w,\beta) + \frac{C_3}{\rho}I(w,\beta) + \frac{C_3}{\rho}J(w,\beta)$$
$$+C_3\rho^7\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7\chi|w|^2dxdt$$
$$\text{for} \quad (\rho,\mu,\nu) \in (0,\infty)\times[1,\infty)\times[T^2,\infty).$$

Therefore, adding (3.22) with the previous two inequalities we obtain

$$\iint_{Q_{\omega_1}} \left(\nu^5\mu^6\beta^5\chi|w_x|^2 + \nu^3\mu^4\beta^3\chi|w_{xx}|^2 + \nu\mu^2\beta\chi|w_{xxx}|^2\right)dxdt$$
$$\leq \frac{C_3\rho^3}{\mu^2}I(w,\beta)$$
$$+\frac{C_3}{\rho}I(w,\beta) + \frac{C_3}{\rho}J(w,\beta) + C_3\rho^7\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2dxdt$$
$$\text{for} \quad (\rho,\mu,\nu) \in [1,\infty)\times[1,\infty)\times[T^2,\infty). \tag{3.27}$$

Finally, since

$$\iint_{Q_{\omega_0}} \left(\nu^5\mu^6\beta^5|w_x|^2 + \nu^3\mu^4\beta^3|w_{xx}|^2 + \nu\mu^2\beta|w_{xxx}|^2\right)dxdt$$
$$\leq \iint_{Q_{\omega_1}} \left(\nu^5\mu^6\beta^5\chi|w_x|^2 + \nu^3\mu^4\beta^3\chi|w_{xx}|^2 + \nu\mu^2\beta\chi|w_{xxx}|^2\right)dxdt,$$

from (3.18) and (3.27) we arrive at

$$M(w) \geq \left(C_2 - \frac{1}{\mu} - \frac{C_3\rho^3}{\mu^2} - \frac{C_3}{\rho}\right)I(w,\beta) - \frac{C_3}{\rho}J(w,\beta)$$
$$-C_3\rho^7\iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2dxdt$$
$$\text{for} \quad (\rho,\mu,\nu) \in [1,\infty)\times[1,\infty)\times[T^2,\infty). \tag{3.28}$$

Part 4 In this part we handle the rest terms defined in $\mathcal{P}_3 w$ and (3.16). We pay special attention to the first term in $\mathcal{P}_3 w$ and to the first two terms of the right-hand side of (3.16). Using that $|\alpha_t| \leq C_4 T\beta^2$, $|\alpha_x| \leq C_4\mu\beta$ and $|\alpha_{xt}| \leq C_4 T\beta^2$, then applying Cauchy's inequality and finally employing that $\mu \leq C_4 T^2\beta$ we can get the following inequalities.

$$\nu^2 \iint_Q \alpha_t^2 |w|^2 dxdt \leq \frac{C_4}{\nu^5} \left(T^2\right)^3 T^2 \frac{1}{\mu^{11}} \iint_Q \nu^7 \mu^8 \beta^7 |w|^2 dxdt.$$

$$\frac{\nu^4}{2} \iint_Q \left(\alpha_x^4\right)_t |w|^2 dxdt = 2\nu^4 \iint_Q \alpha_x^3 \alpha_{xt} |w|^2 dxdt$$
$$\leq \frac{C_4}{\nu^3} \left(T^2\right)^2 T \frac{1}{\mu^6} \iint_Q \nu^7 \mu^8 \beta^7 |w|^2 dxdt.$$

$$3\nu^2 \iint_Q \left(\alpha_x^2\right)_t |w_x|^2 dxdt \leq 6\nu^2 \iint_Q \alpha_x \alpha_{xt} |w_x|^2 dxdt$$
$$\leq \frac{C_4}{\nu^3} \left(T^2\right)^2 T \frac{1}{\mu^6} \iint_Q \nu^5 \mu^6 \beta^5 |w_x|^2 dxdt.$$

The remaining terms in $\mathcal{P}_3 w$ and (3.16) can be handled as we just did. Accordingly, the combination of the previous three inequalities yield

$$\|\mathcal{P}_3 w\|^2_{L^2(Q)} + |R(w)| \leq \frac{C_4}{\mu} I(w, \beta) \quad \text{for} \quad (\mu, \nu) \in [1, \infty) \times [\max\{T, T^2\}, \infty). \tag{3.29}$$

Let us note that the inequalities obtained here made us to ask $\nu \geq \max\{T, T^2\}$ instead of $\nu \geq T^2$, which was the choice in the previous parts of the proof.

Part 5 In this part we derive the two Carleman estimates mentioned at the beginning of the proof. Combining (3.8), (3.13), (3.28) and (3.29) it follows

$$\frac{1}{2}\|\mathcal{P}_1 w\|^2_{L^2(Q)} + \frac{1}{2}\|\mathcal{P}_2 w\|^2_{L^2(Q)} + \left(C_2 - \frac{1}{\mu} - \frac{C_3\rho^3}{\mu^2} - \frac{C_3}{\rho}\right) I(w, \beta)$$
$$- \frac{C_3}{\rho} J(w, \beta) + B(w, \alpha, L) - B(w, \alpha, 0) \leq \|\mathcal{P} w\|^2_{L^2(Q)}$$
$$+ C_3 \rho^7 \iint_{Q_{\omega_1}} \nu^7 \mu^8 \beta^7 |w|^2 dxdt$$
$$\text{for} \quad (\rho, \mu, \nu) \in [1, \infty) \times [1, \infty) \times [\max\{T, T^2\}, \infty).$$

Then, choosing $\rho_0 \geq 1$ large enough gives us

$$\frac{1}{2}\|\mathcal{P}_1 w\|^2_{L^2(Q)} + \frac{1}{2}\|\mathcal{P}_2 w\|^2_{L^2(Q)} + \frac{C_2}{2} I(w, \beta) - \frac{C_3}{\rho} J(w, \beta)$$
$$+ B(w, \alpha, L) - B(w, \alpha, 0) \leq \|\mathcal{P} w\|^2_{L^2(Q)} + C_3 \rho^7 \iint_{Q_{\omega_1}} \nu^7 \mu^8 \beta^7 |w|^2 dxdt$$
$$\text{for} \quad (\rho, \mu, \nu) \in [\rho_0, \infty) \times [\rho^3, \infty) \times [\max\{T, T^2\}, \infty).$$

Here taking into account (3.11) yields

$$
\begin{aligned}
&\frac{\min\{\frac{1}{2},\frac{C_2}{4}\}}{C_1} J(w,\beta) + \frac{C_2}{4} I(w,\beta) - \frac{C_3}{\rho} J(w,\beta) \\
&\quad + B(w,\alpha,L) - B(w,\alpha,0) \le \|Pw\|^2_{L^2(Q)} + C_3\rho^7 \iint_{Q_{\omega_1}} \nu^7\mu^8\beta^7|w|^2 dxdt \\
&\quad \text{for} \quad (\rho,\mu,\nu) \in [\rho_0,\infty) \times [\rho^3,\infty) \times [\max\{T,T^2\},\infty).
\end{aligned}
$$

Recalling that $\mathcal{P}w = e^{-\nu\alpha}\mathcal{L}^*(e^{\nu\alpha}w)$ and $w = e^{-\nu\alpha}q$, let us set $\rho = \rho_0$ and then choose $\rho_0 \ge 1$ large enough to obtain

$$
\begin{aligned}
&C_5 I(w,\beta) + C_5 J(w,\beta) + B(w,\alpha,L) - B(w,\alpha,0) \le \iint_Q e^{-2\nu\alpha}|\mathcal{L}^* q|^2 dxdt \\
&\quad + C_6 \iint_{Q_{\omega_1}} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2 dxdt \\
&\quad \text{for} \quad (\mu,\nu) \in [\rho_0^3,\infty) \times [\max\{T,T^2\},\infty).
\end{aligned}
\tag{3.30}
$$

Let us note that the boundary terms $B(w,\alpha,L)$ and $B(w,\alpha,0)$ in the left-hand side of (3.30) do not allow us to derive Carleman estimate (3.7). To overcome this difficulty we are going to adapt the ideas used in [20, Chapter I, Lemma 1.2], which consists in deriving a second Carleman estimate, with similar weight functions, whose boundary terms will cancel the ones of (3.30) when adding.

Recalling that $\psi \in C^4([0,L])$ satisfies (3.1)–(3.3), for $\mu > 0$ let us define the weight functions

$$
\widetilde{\alpha}(t,x) = \frac{e^{4\mu\|\psi\|_{L^\infty(0,L)}}}{t(T-t)} - \widetilde{\beta}(t,x), \quad \widetilde{\beta}(t,x) = \frac{e^{2\mu\|\psi\|_{L^\infty(0,L)} - \mu\psi(x)}}{t(T-t)}, \quad (t,x) \in \overline{Q},
\tag{3.31}
$$

which are slightly different to the ones defined in (3.4). These weight functions also satisfy the asymptotic behavior of the weight functions defined in (3.4), which is

$$
\lim_{t\to 0^+} \widetilde{\alpha}(t,\cdot) = \lim_{t\to T^-} \widetilde{\alpha}(t,\cdot) = \lim_{t\to 0^+} \widetilde{\beta}(t,\cdot) = \lim_{t\to T^-} \widetilde{\beta}(t,\cdot) = \infty.
\tag{3.32}
$$

Also, we have

$$
\left.
\begin{aligned}
&\widetilde{\alpha}_x = \mu\psi'\widetilde{\beta}, \\
&\widetilde{\alpha}_{xx} = \mu\psi''\widetilde{\beta} - \mu^2\left(\psi'\right)^2\widetilde{\beta}, \\
&\widetilde{\alpha}_{xxx} = \mu\psi'''\widetilde{\beta} - 3\mu^2\psi'\psi''\widetilde{\beta} + \mu^3\left(\psi'\right)^3\widetilde{\beta}.
\end{aligned}
\right\}
\tag{3.33}
$$

Let us consider $\widetilde{w} = e^{-\nu\widetilde{\alpha}}q$. In view of (3.31)–(3.33), we see that the same arguments used until now allow us to obtain $\widetilde{\rho}_0 \geq 1$ large enough such that

$$\widetilde{C}_5 I(\widetilde{w}, \widetilde{\beta}) + \widetilde{C}_5 J(\widetilde{w}, \widetilde{\beta}) + B(\widetilde{w}, \widetilde{\alpha}, L) - B(\widetilde{w}, \widetilde{\alpha}, 0) \leq \iint_Q e^{-2\nu\widetilde{\alpha}}|\mathcal{L}^* q|^2 dxdt$$
$$+ \widetilde{C}_6 \iint_{Q_{\omega_1}} e^{-2\nu\widetilde{\alpha}} \nu^7 \mu^8 \widetilde{\beta}^7 |q|^2 dxdt$$
$$\text{for} \quad (\mu, \nu) \in [\widetilde{\rho}_0{}^3, \infty) \times [\max\{T, T^2\}, \infty). \tag{3.34}$$

To finish this part let us prove

$$B(w, \alpha, 0) + B(\widetilde{w}, \widetilde{\alpha}, 0) = B(w, \alpha, L) + B(\widetilde{w}, \widetilde{\alpha}, L) = 0. \tag{3.35}$$

Since $q_x(\cdot, 0) = q_x(\cdot, L) = 0$ and $q_{xxx}(\cdot, 0) = q_{xxx}(\cdot, L) = 0$, we see that (3.2) together with (3.4), (3.17), (3.31) and (3.33) imply that $\alpha(\cdot, x) = \widetilde{\alpha}(\cdot, x)$, $\alpha_x(\cdot, x) = -\widetilde{\alpha}_x(\cdot, x)$, $\alpha_{xx}(\cdot, x) = \widetilde{\alpha}_{xx}(\cdot, x)$ and $\alpha_{xxx}(\cdot, x) = -\widetilde{\alpha}_{xxx}(\cdot, x)$ for $x \in \{0, L\}$. Furthermore, with $w = e^{-\nu\alpha}q$ and $\widetilde{w} = e^{-\nu\widetilde{\alpha}}q$ in mind, we get that $w(\cdot, x) = \widetilde{w}(\cdot, x)$, $w_x(\cdot, x) = -\widetilde{w}_x(\cdot, x)$, $w_{xx}(\cdot, x) = \widetilde{w}_{xx}(\cdot, x)$ and $w_{xxx}(\cdot, x) = -\widetilde{w}_{xxx}(\cdot, x)$ for $x \in \{0, L\}$. Therefore, recalling (3.15) we can deduce (3.35).

Part 6 In this part we derive Carleman estimate (3.7). Taking into account (3.30) and (3.34), let us set $\mu_0 = \max\{\rho_0^3, \widetilde{\rho}_0{}^3\} \geq 1$. Then, from (3.30), (3.34) and (3.35) we have

$$I(w, \beta) + J(w, \beta) + I(\widetilde{w}, \widetilde{\beta}) + J(\widetilde{w}, \widetilde{\beta}) \leq C \iint_Q \left(e^{-2\nu\alpha} + e^{-2\nu\widetilde{\alpha}}\right) |\mathcal{L}^* q|^2 dxdt$$
$$+ C \iint_{Q_{\omega_1}} e^{-2\nu\alpha} \nu^7 \mu^8 \beta^7 |q|^2 dxdt$$
$$+ C \iint_{Q_{\omega_1}} e^{-2\nu\widetilde{\alpha}} \nu^7 \mu^8 \widetilde{\beta}^7 |q|^2 dxdt$$
$$\text{for} \quad (\mu, \nu) \in [\mu_0, \infty) \times [\max\{T, T^2\}, \infty).$$

Here we employ that $\beta \geq \widetilde{\beta}$ to obtain

$$I(w, \beta) + J(w, \beta) \leq C \iint_Q e^{-2\nu\alpha}|\mathcal{L}^* q|^2 dxdt + C \iint_{Q_{\omega_1}} e^{-2\nu\alpha} \nu^7 \mu^8 \beta^7 |q|^2 dxdt$$
$$\text{for} \quad (\mu, \nu) \in [\mu_0, \infty) \times [\max\{T, T^2\}, \infty). \tag{3.36}$$

Let us recall that $I(w, \beta)$ and $J(w, \beta)$ were respectively defined in (3.9) and (3.10). Finally, Carleman estimate (3.7) is a consequence of (3.36), $w = e^{-\nu\alpha}q$ and the inequalities

$$\iint_Q e^{-2\nu\alpha}\frac{1}{\nu\beta}\left(|(e^{\nu\alpha}w)_t|^2+|(e^{\nu\alpha}w)_{xxxx}|^2\right)dxdt \le CI(w,\beta)+CJ(w,\beta),$$

$$\sum_{n=0}^{3}\iint_Q e^{-2\nu\alpha}\nu^{7-2n}\mu^{8-2n}\beta^{7-2n}\left|\frac{\partial^n(e^{\nu\alpha}w)}{\partial x^n}\right|^2 dxdt \le CI(w,\beta).$$

The proof of Proposition 3.1 is complete. □

4 Control Results

In this section we prove our main result, which is the one given by Theorem 1.1. With the weight functions introduced in Sect. 3.1 in mind, for $\mu > 0$ let us define

$$M = \frac{T^2}{4}\max_{x\in[0,L]}\alpha(T/2,x) = e^{4\mu\|\psi\|_{L^\infty(0,L)}} - e^{2\mu\|\psi\|_{L^\infty(0,L)}}, \tag{4.1}$$

$$m = \frac{T^2}{4}\min_{x\in[0,L]}\alpha(T/2,x) = e^{4\mu\|\psi\|_{L^\infty(0,L)}} - e^{3\mu\|\psi\|_{L^\infty(0,L)}}. \tag{4.2}$$

The results in this section require that $3m > 2M$. Then, in virtue of the inequality $e^x \ge 1 + x$ for $x \ge 0$, we see from (4.1) and (4.2) that it suffices to ask that $\mu \ge 2/\|\psi\|_{L^\infty(0,L)}$. Here we also require to apply Carleman estimate (3.7). Therefore, from now on we employ

$$\mu = \max\{2/\|\psi\|_{L^\infty(0,L)}, \mu_0\} \quad \text{and} \quad \nu = \max\{T, T^2\}. \tag{4.3}$$

Finally, let us introduce $\phi \in C^1([0,T])$ by

$$\phi(t) = \begin{cases} 4/T^2 & , 0 \le t \le T/2, \\ \dfrac{1}{t(T-t)} & , T/2 \le t \le T. \end{cases} \tag{4.4}$$

4.1 Linear Equation

In this section we prove the null controllability of (1.8). Let us begin with some preliminary results. The first two preliminary results are estimates for the operator $\mathcal{L}^*q = -q_t + q_{xxxx} + A_2q_{xx} + A_1q_x + A_0q$, with $A_i \in L^\infty(0,T;L^\infty(0,L))$ for $i \in \{0,1,2\}$ and $q \in \mathcal{Q}$, which was defined in (3.6).

Proposition 4.1 *There exist $C > 0$ such that for any $q \in \mathcal{Q}$ we have*

$$\iint_Q e^{-2\nu M\phi}\phi^7|q|^2dxdt + \int_0^L |q(0,x)|^2dx \le C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^*q|^2dxdt + C\iint_{Q_\omega} e^{-2\nu m\phi}\phi^7|q|^2dxdt. \tag{4.5}$$

Proof Let us take $\chi \in C^1([0,T])$ such that $\chi = 1$ in $[0, T/2]$ and $\chi = 0$ in $[3T/4, T]$. Since $\chi q \in \mathcal{Q}$, $\mathcal{L}^*(\chi q) = \chi \mathcal{L}^* q - \chi' q$ and $\chi(T)q(T,\cdot) = 0$, it follows that the change of variable $t \to T - t$ allows us to apply (2.3) to obtain

$$\|\chi q\|^2_{L^\infty(0,T;L^2(0,L))} \le C\|\chi \mathcal{L}^* q - \chi' q\|^2_{L^2(0,T;L^2(0,L))},$$

from which it follows

$$\|q\|^2_{L^\infty(0,T/2;L^2(0,L))} \le C\|\mathcal{L}^* q\|^2_{L^2(0,3T/4;L^2(0,L))} + C\|q\|^2_{L^2(T/2,3T/4;L^2(0,L))}. \tag{4.6}$$

Taking into account that $e^{-2\nu M\phi}\phi^7 \le C$ in $[0, T/2]$, we have that (4.6) implies

$$\int_0^{T/2}\int_0^L e^{-2\nu M\phi}\phi^7|q|^2dxdt + \int_0^L |q(0,x)|^2dx$$
$$\le C\|q\|^2_{L^\infty(0,T/2;L^2(0,L))} \le C\int_0^{3T/4}\int_0^L |\mathcal{L}^* q|^2dxdt + C\int_{T/2}^{3T/4}\int_0^L |q|^2dxdt.$$

Here using that $0 < C \le e^{-2\nu m\phi}$ in $[0, 3T/4]$ and $0 < C \le e^{-2\nu\alpha}\nu^7\mu^8\beta^7$ in $[T/2, 3T/4] \times [0, L]$ yield

$$\int_0^{T/2}\int_0^L e^{-2\nu M\phi}\phi^7|q|^2dxdt + \int_0^L |q(0,x)|^2dx$$
$$\le C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^* q|^2dxdt + C\iint_Q e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt. \tag{4.7}$$

Applying Carleman estimate (3.7) with (4.3) to the last term of the right-hand side of (4.7) and then using that $\alpha \ge m\phi$ we get

$$\int_0^{T/2}\int_0^L e^{-2\nu M\phi}\phi^7|q|^2dxdt + \int_0^L |q(0,x)|^2dx$$
$$\le C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^* q|^2dxdt + C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt. \tag{4.8}$$

Thanks to Carleman estimate (3.7) with (4.3) it follows

$$\int_{T/2}^T\int_0^L e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt \le C\iint_Q e^{-2\nu\alpha}|\mathcal{L}^* q|^2dxdt$$
$$+ C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt.$$

Here considering that $0 < C\phi^7 \le \nu^7\mu^8\beta^7$ in $[T/2, T]\times[0, L]$, $M\phi \ge \alpha$ in $[T/2, T]\times [0, L]$ and $\alpha \ge m\phi$ we arrive at

$$\int_{T/2}^{T}\int_0^L e^{-2\nu M\phi}\phi^7|q|^2dxdt \leq C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^*q|^2dxdt + C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt,$$

which combined with (4.8) leads us to

$$\iint_Q e^{-2\nu M\phi}\phi^7|q|^2dxdt + \int_0^L |q(0,x)|^2dx \leq C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^*q|^2dxdt + C\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt. \quad (4.9)$$

It only left to handle the last term of the right-hand side of (4.9). In order to do so we take into account that $e^{-2\nu\alpha}\nu^7\mu^8\beta^7 \leq C$ in $[0,T/2]\times[0,L]$, $0 < C \leq e^{-2\nu m\phi}\phi^7$ in $[0,T/2]$, $\alpha \geq m\phi$ and $\beta \leq C\phi$ in $[T/2,T]\times[0,L]$ to obtain

$$\begin{aligned}\iint_{Q_\omega} e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt &= \int_0^{T/2}\int_\omega e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt \\ &\quad + \int_{T/2}^{T}\int_\omega e^{-2\nu\alpha}\nu^7\mu^8\beta^7|q|^2dxdt \\ &\leq C\int_0^{T/2}\int_\omega |q|^2dxdt \\ &\quad + C\int_{T/2}^{T}\int_\omega e^{-2\nu m\phi}\phi^7|q|^2dxdt \\ &\leq C\iint_{Q_\omega} e^{-2\nu m\phi}\phi^7|q|^2dxdt.\end{aligned}$$

Finally, we have that (4.5) is consequence of (4.9) together with the previous inequality. The proof of Proposition 4.1 is complete. □

Corollary 4.1 *There exist $C > 0$ such that for any $q \in \mathcal{Q}$ we have*

$$\left\|e^{-\nu M\phi}\phi^{3/2}q\right\|_{L^\infty(0,T;L^2(0,L))} + \|q(0,\cdot)\|_{L^2(0,L)} \leq C\left\|e^{-\nu m\phi}\mathcal{L}^*q\right\|_{L^2(0,T;L^2(0,L))} + C\left\|e^{-\nu m\phi}\phi^{7/2}q\right\|_{L^2(0,T;L^2(\omega))}. \quad (4.10)$$

Proof For $t \in [0,T]$ let us introduce $\chi(t) = e^{-\nu M\phi(t)}\phi(t)^{3/2}$. Then, keeping in mind that $\chi q \in \mathcal{Q}$, $\mathcal{L}^*(\chi q) = \chi\mathcal{L}^*q - \chi' q$ and $\chi(T)q(T,\cdot) = 0$, the change of variable $t \to T - t$ allows us to apply (2.3) to obtain

$$\|\chi q\|^2_{L^\infty(0,T;L^2(0,L))} \leq C\|\chi\mathcal{L}^*q\|^2_{L^2(0,T;L^2(0,L))} + C\|\chi' q\|^2_{L^2(0,T;L^2(0,L))}. \quad (4.11)$$

On the one hand, since by the definition of (4.1) and (4.2) we have that $M - m > 0$, we can employ that $e^{-2\nu(M-m)\phi}\phi^3 \leq C$ to get

$$\begin{aligned} \|\chi\mathcal{L}^*q\|^2_{L^2(0,T;L^2(0,L))} &= \iint_Q e^{-2\nu M\phi}\phi^3|\mathcal{L}^*q|^2 dxdt \\ &= \iint_Q e^{-2\nu m\phi}\left(e^{-2\nu(M-m)\phi}\phi^3\right)|\mathcal{L}^*q|^2 dxdt \\ &\leq C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^*q|^2 dxdt. \end{aligned} \tag{4.12}$$

On the other hand, in view of $|\chi'| \leq Ce^{-\nu M\phi}\left(\phi^{7/2} + \phi^{5/2}\right)$ and $\phi^{-1} \leq T^2/4$, we see that (4.11) and (4.12) lead us to

$$\|\chi q\|^2_{L^\infty(0,T;L^2(0,L))} \leq C\iint_Q e^{-2\nu m\phi}|\mathcal{L}^*q|^2 dxdt + C\iint_Q e^{-2\nu M\phi}\phi^7|q|^2 dxdt.$$

Finally, we have that (4.10) is consequence of (4.5) together with the previous inequality. The proof of Corollary 4.1 is complete. □

To prove the null controllability of (1.8) we require to set up some notation. Taking into account (4.1), (4.2), (4.3) and (4.4), let us define the weighted Banach spaces $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$, $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$ and $\left(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}}\right)$ as follows.

$$\mathcal{F} = \left\{f \;/\; e^{\nu M\phi}\phi^{-3/2}f \in L^1(0,T;L^2(0,L)\right\}, \tag{4.13}$$

endowed with the norm $\|f\|_{\mathcal{F}} = \|e^{\nu M\phi}\phi^{-3/2}f\|_{L^1(0,T;L^2(0,L))}$.

$$\mathcal{U} = \left\{u \;/\; e^{\nu m\phi}\phi^{-7/2}u \in L^2(0,T;L^2(\omega))\right\}, \tag{4.14}$$

endowed with the norm $\|u\|_{\mathcal{U}} = \|e^{\nu m\phi}\phi^{-7/2}u\|_{L^2(0,T;L^2(\omega))}$.

$$\mathcal{Y} = \left\{y \;/\; e^{\nu m\phi}\phi^{-7/2}y \in C([0,T];L^2(0,L)) \cap L^2(0,T;\mathcal{H}_2)\right\}, \tag{4.15}$$

endowed with the norm $\|y\|_{\mathcal{Y}} = \|e^{\nu m\phi}\phi^{-7/2}y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H^2(0,L))}$.

Theorem 4.1 *Let $\overline{z} \in L^\infty(0,T;W^{2,\infty}(0,L))$. Then, for every $(f, y_0) \in \mathcal{F}\times L^2(0,L)$ there exists a control $u \in \mathcal{U}$ such that (1.8) has a unique solution $y \in \mathcal{Y}$.*

Remark 4.1 This theorem establishes the null controllability of (1.8). Indeed, this follows from the fact that $e^{\nu m\phi}\phi^{-7/2}y \in C([0,T];L^2(0,L))$ implies that $y(T,\cdot) = 0$.

Proof of Theorem 4.1 The proof is quite similar to the proof of [18, Proposition 2], which follows the ideas developed in [20,26]. Let $a_2 = -3\overline{z}^2 + 1$, $a_1 = -12\overline{z}\overline{z}_x$ and

$a_0 = -6\overline{z}\overline{z}_{xx} - 6\overline{z}_x^2$. Let us consider the formal adjoint operator of the operator

$$\mathcal{L}y = y_t + y_{xxxx} + \sum_{k=0}^{2} a_k \frac{\partial^k y}{\partial x^k},$$

which is

$$\mathcal{L}^* q = -q_t + q_{xxxx} + \sum_{k=0}^{2} (-1)^k \frac{\partial^k}{\partial x^k} (a_k q) = -q_t + q_{xxxx} + A_2 q_{xx} + A_1 q_x + A_0 q,$$

where $A_2 = a_2$, $A_1 = 2\,(a_2)_x - a_1$ and $A_0 = (a_2)_{xx} - (a_1)_x + a_0$ are elements of $L^\infty(0, T; L^\infty(0, L))$ because $\overline{z} \in L^\infty(0, T; W^{2,\infty}(0, L))$.

Recalling that $\mathcal{Q}$ was defined in (3.6), let us consider the symmetric bilinear form $a(\cdot, \cdot) : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ given by

$$a\left(\hat{q}, q\right) = \iint_{Q} e^{-2\nu m\phi} \left(\mathcal{L}^* \hat{q}\right) \left(\mathcal{L}^* q\right) dx dt + \iint_{Q_\omega} e^{-2\nu m\phi} \phi^7 \hat{q} q \, dx dt. \tag{4.16}$$

Since (4.10) implies that $q \in \mathcal{Q} \mapsto a\,(q, q)^{1/2} \in [0, \infty)$ is a norm for $\mathcal{Q}$, let us introduce $\overline{\mathcal{Q}}$ as the completion of $\mathcal{Q}$ induced by such a norm. Note that $\overline{\mathcal{Q}}$ is a Hilbert space for the scalar product $a(\cdot, \cdot)$. Therefore, the symmetric bilinear form $a(\cdot, \cdot)$ is well defined, continuous and coercive on $\overline{\mathcal{Q}}$. Let us also consider the linear form $l : \mathcal{Q} \to \mathbb{R}$ given by

$$l(q) = \langle f, q \rangle_{L^1(0,T;L^2(0,L)) \times L^\infty(0,T;L^2(0,L))} + \int_0^L y_0(x) q(0, x) \, dx. \tag{4.17}$$

Here we have adopted the notation $\langle \cdot, \cdot \rangle_{X \times X^*}$ to denote the duality product between a Banach space X and its dual space X^*. Thanks to the Cauchy–Schwarz inequality and (4.10) we arrive at

$$|l(q)| \le C \left(\| e^{\nu M\phi} \phi^{-3/2} f \|_{L^1(0,T;L^2(0,L))} + \| y_0 \|_{L^2(0,L)} \right) a\,(q, q)^{1/2}. \tag{4.18}$$

Since $(f, y_0) \in \mathcal{F} \times L^2(0, L)$, by (4.18) we conclude that the linear form $l(\cdot)$ is well defined and continuous on $\overline{\mathcal{Q}}$. Therefore, the Lax–Milgram lemma tells us that the variational problem

$$a\left(\hat{q}, q\right) = l(q) \quad \text{for every} \quad q \in \overline{\mathcal{Q}}. \tag{4.19}$$

has a unique solution $\hat{q} \in \overline{\mathcal{Q}}$.

Let us define

$$y = e^{-2\nu m\phi} \left(\mathcal{L}^* \hat{q}\right) \quad \text{and} \quad u = -e^{-2\nu m\phi} \phi^7 \hat{q} \mathbb{1}_\omega. \tag{4.20}$$

Then, in virtue of (4.16) and (4.18)–(4.20) it follows

$$\|e^{\nu m\phi} y\|_{L^2(0,T;L^2(0,L))} + \|e^{\nu m\phi}\phi^{-7/2} u\|_{L^2(0,T;L^2(\omega))} \le C\|f\|_{\mathcal{F}} + C\|y_0\|_{L^2(0,L)}. \tag{4.21}$$

Noting that (4.21) allows us to conclude that $e^{\nu m\phi} y \in L^2(0,T;L^2(0,L))$ and $u \in \mathcal{U}$, thanks to (4.16), (4.17), (4.19) and (4.20) we deduce that $y \in L^2(0,T;L^2(0,L))$ is a solution defined by transposition of (1.8), that is to say, for every $g \in L^2(0,T;L^2(0,L))$ we have

$$\iint_Q yg\,dxdt = \langle f, q\rangle_{L^1(0,T;L^2(0,L))\times L^\infty(0,T;L^2(0,L))} + \iint_{Q_\omega} uq\,dxdt + \int_0^L y_0(x)q(0,x),$$

where $q \in C([0,T];L^2(0,L)) \cap L^2(0,T;\mathcal{H}_2)$ is the unique solution given by Proposition 2.1 of

$$\begin{cases} \mathcal{L}^* q = g, & (t,x) \in (0,T)\times(0,L), \\ q_x(t,0) = 0, q_x(t,L) = 0, & t \in (0,T), \\ q_{xxx}(t,0) = 0, q_{xxx}(t,L) = 0, & t \in (0,T), \\ q(T,x) = 0, & x \in (0,L). \end{cases}$$

Now it only left to prove that $e^{\nu m\phi}\phi^{-7/2} y \in C([0,T];L^2(0,L)) \cap L^2(0,T;\mathcal{H}_2)$. For $t \in [0,T]$ let us consider $\chi(t) = e^{\nu m\phi(t)}\phi(t)^{-7/2}$. Then, in view of (1.8) we see that χy satisfies

$$\begin{cases} \mathcal{L}(\chi y) = \chi f + \chi \mathbb{1}_\omega u - \chi' y, & (t,x) \in (0,T)\times(0,L), \\ (\chi y)_x(t,0) = 0, (\chi y)_x(t,L) = 0, & t \in (0,T), \\ (\chi y)_{xxx}(t,0) = 0, (\chi y)_{xxx}(t,L) = 0, & t \in (0,T), \\ (\chi y)(0,x) = \chi(0) y_0(x), & x \in (0,L). \end{cases} \tag{4.22}$$

Taking into account that $|\chi'| \le Ce^{\nu m\phi}\left(\phi^{-3/2} + \phi^{-5/2}\right)$ and $\phi^{-1} \le T^2/4$, we can get the following inequalities.

$$\begin{aligned} \|\chi f\|_{L^1(0,T;L^2(0,L))} &\le C\|e^{\nu M\phi}\phi^{-3/2} f\|_{L^1(0,T;L^2(0,L))}, \\ \|\chi \mathbb{1}_\omega u\|_{L^1(0,T;L^2(0,L))} &\le C\|e^{\nu m\phi}\phi^{-7/2} u\|_{L^2(0,T;L^2(\omega))}, \\ \|\chi' y\|_{L^1(0,T;L^2(0,L))} &\le C\|e^{\nu m\phi} y\|_{L^2(0,T;L^2(0,L))}. \end{aligned}$$

Finally, since $f \in \mathcal{F}$, $u \in \mathcal{U}$ and $e^{\nu m\phi} y \in L^2(0,T;L^2(0,L))$, Proposition 2.1 applied to (4.22) allows us to conclude that $y \in \mathcal{Y}$. The proof of Theorem 4.1 is complete. □

4.2 Nonlinear Equation

As we did in the previous section, let us begin with some preliminary results. The first of them is an interpolation inequality.

Lemma 4.1 *If* $y \in C([0, T]; L^2(0, L)) \cap L^2(0, T; H^2(0, L))$*, then* $y \in L^4(0, T; H^1(0, L))$. *Moreover, there exists* $C > 0$ *such that*

$$\|y\|_{L^4(0,T;H^1(0,L))} \leq C\|y\|^{1/2}_{C([0,T];L^2(0,L))}\|y\|^{1/2}_{L^2(0,T;H^2(0,L))}. \tag{4.23}$$

Proof Since $y(t, \cdot) \in L^2(0, L)$ for every $t \in [0, T]$ and $y(t, \cdot) \in H^2(0, L)$ for almost every $t \in [0, T]$, thanks to a convexity inequality for $H^2(0, L)$, which may be found in [14, Chapter IV, Part 7, Section 3, Proposition 4], we have

$$\|y(t, \cdot)\|^4_{H^1(0,L)} \leq C\|y(t, \cdot)\|^2_{L^2(0,L)}\|y(t, \cdot)\|^2_{H^2(0,L)} \quad \text{for almost every} \quad t \in [0, T].$$

Then, the previous inequality implies (4.23). The proof of Lemma 4.1 is complete. □

The second preliminary result corresponds to the continuity of some multilinear forms that later will be required for the proof of Theorem 1.1.

Lemma 4.2 *Let* $b_i \in L^\infty(0, T; L^\infty(0, L))$ *for* $i \in \{0, 1, 2, 3, 4, 5\}$. *Let us consider the bilinear and trilinear forms* $\mathcal{N}_{Bi} : \mathcal{Y} \times \mathcal{Y} \to \mathcal{F}$ *and* $\mathcal{N}_{Tri} : \mathcal{Y} \times \mathcal{Y} \times \mathcal{Y} \to \mathcal{F}$ *respectively given by*

$$\mathcal{N}_{Bi}(y_1, y_2) = b_5 y_1 y_2 + b_4 (y_1)_x (y_2)_x + b_3 y_1 (y_2)_x + b_2 y_1 (y_2)_{xx}, \tag{4.24}$$

$$\mathcal{N}_{Tri}(y_1, y_2, y_3) = b_1 y_1 (y_2)_x (y_3)_x + b_0 y_1 y_2 (y_3)_{xx}. \tag{4.25}$$

Then, there exists $C > 0$ *such that*

$$\|\mathcal{N}_{Bi}(y_1, y_2)\|_{\mathcal{F}} \leq C\|y_1\|_{\mathcal{Y}}\|y_2\|_{\mathcal{Y}}, \tag{4.26}$$

$$\|\mathcal{N}_{Tri}(y_1, y_2, y_3)\|_{\mathcal{F}} \leq C\|y_1\|_{\mathcal{Y}}\|y_2\|_{\mathcal{Y}}\|y_3\|_{\mathcal{Y}}. \tag{4.27}$$

Proof Let us recall that $\|\cdot\|_{\mathcal{F}}$ and $\|\cdot\|_{\mathcal{Y}}$ were respectively defined in (4.13) and (4.15). We use the fact that $3m > 2M$, which is guaranteed due to (4.3), and the inequality $e^x \geq x^n/n!$ for $x \geq 0$ and $n \in \mathbb{N}$ to deduce

$$e^{\nu M\phi}\phi^{-3/2} \leq \left(\nu^{-n}m^{-n}n!\right)^{1/2} e^{2\nu m\phi}\phi^{-(n+3)/2}. \tag{4.28}$$

In what follows we are going to use several times (4.28) for different values of $n \in \mathbb{N}$, the continuous injection $H^1(0, L) \hookrightarrow L^\infty(0, L)$ and Hölder's inequality. In order to ease the notation we employ $L^2(0, T; L^2(0, L)) = L^2(L^2)$, $L^p(0, T; H^q(0, L)) = L^p(H^q)$ for $(p, q) \in \mathbb{N}^2$ and $C([0, T]; L^2(0, L)) = C(L^2)$.

Let us prove the continuity of $\mathcal{N}_{Bi}$. From (4.24) we have

$$\begin{aligned}\|\mathcal{N}_{Bi}(y_1, y_2)\|_{\mathcal{F}} \le C \int_0^T & e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{H^1(0,L)}\|y_2(t,\cdot)\|_{L^2(0,L)}\,dt\\ &+ C\int_0^T e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{H^2(0,L)}\|y_2(t,\cdot)\|_{H^1(0,L)}\,dt\\ &+ C\int_0^T e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{H^1(0,L)}\|y_2(t,\cdot)\|_{H^1(0,L)}\,dt\\ &+ C\int_0^T e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{H^1(0,L)}\|y_2(t,\cdot)\|_{H^2(0,L)}\,dt,\end{aligned}$$

which leads us to

$$\begin{aligned}\|\mathcal{N}_{Bi}(y_1, y_2)\|_{\mathcal{F}} \le C&\|e^{\nu m\phi}\phi^{-7/2}y_1\|_{L^2(H^1)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^2(L^2)}\\ &+ C\|e^{\nu m\phi}\phi^{-7/2}y_1\|_{L^2(H^2)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^2(H^1)}\\ &+ C\|e^{\nu m\phi}\phi^{-7/2}y_1\|_{L^2(H^1)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^2(H^1)}\\ &+ C\|e^{\nu m\phi}\phi^{-7/2}y_1\|_{L^2(H^1)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^2(H^2)}.\end{aligned}$$

Then, the previous inequality allows us to obtain (4.26). Now let us prove the continuity of $\mathcal{N}_{Tri}$. From (4.25) we have

$$\begin{aligned}&\|\mathcal{N}_{Tri}(y_1, y_2, y_3)\|_{\mathcal{F}}\\ &\quad\le C\int_0^T e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{L^2(0,L)}\|y_2(t,\cdot)\|_{H^2(0,L)}\|y_3(t,\cdot)\|_{H^2(0,L)}\,dt\\ &\qquad+ C\int_0^T e^{\nu M\phi}\phi^{-3/2}\|y_1(t,\cdot)\|_{H^1(0,L)}\|y_2(t,\cdot)\|_{H^1(0,L)}\|y_3(t,\cdot)\|_{H^2(0,L)}\,dt,\end{aligned}$$

which leads us to

$$\begin{aligned}&\|\mathcal{N}_{Tri}(y_1, y_2, y_3)\|_{\mathcal{F}}\\ &\quad\le C\|e^{\nu m\phi}\phi^{-7/2}y_1\|_{C(L^2)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^2(H^2)}\|e^{\nu m\phi}\phi^{-7/2}y_3\|_{L^2(H^2)}\\ &\qquad+ \|e^{\nu m\phi}\phi^{-7/2}y_1\|_{L^4(H^1)}\|e^{\nu m\phi}\phi^{-7/2}y_2\|_{L^4(H^1)}\|e^{\nu m\phi}\phi^{-7/2}y_3\|_{L^2(H^2)}.\end{aligned}$$

Then, the previous inequality together with interpolation inequality (4.23) allow us to obtain (4.27). The proof of Lemma 4.2 is complete. □

The third and last preliminary result corresponds to a local inversion theorem that may be found in [1, Chapter 2, Section 2.3.4] or [20, Chapter I, Section 4, Theorem 4.1].

Theorem 4.2 *Let E and G be Banach spaces. Let $\Lambda \in C^1(E; G)$ and let us assume that there exists $\hat{e} \in E$ such that $\Lambda'(\hat{e}) : E \to G$ is surjective. Then, there exists $\varepsilon > 0$*

such that for every $g \in G$ satisfying that $\|g - \Lambda(\hat{e})\|_G < \varepsilon$ there exists $e \in E$ solving the equation $\Lambda(e) = g$.

Finally, we can prove our main result.

Proof of Theorem 1.1 Let us recall that the operators $\mathcal{L}y$ and $\mathcal{N}y$ were respectively defined in (1.6) and (1.7). Also, let us note that their coefficients are elements of $L^\infty(0, T; L^\infty(0, L))$ because $\overline{z} \in L^\infty(0, T; W^{2,\infty}(0, L))$.

Considering (4.13)–(4.15), let us define the Banach spaces $E = \{(u, y) \in \mathcal{U} \times \mathcal{Y} \,/\, \mathcal{L}y - \mathbb{1}_\omega u \in \mathcal{F}\}$ and $G = \mathcal{F} \times L^2(0, L)$. Let us introduce $\Lambda : E \to G$ by $\Lambda(u, y) = (\mathcal{L}y - \mathbb{1}_\omega u + \mathcal{N}y, y(0, \cdot))$. Thanks to Lemma 4.2 we have that $\Lambda : E \to G$ is well defined and that $\Lambda \in C^1(E; G)$. Indeed:

- By (4.24) and (4.25) we see that $\mathcal{N}y = \mathcal{N}_{Bi}(y, y) + \mathcal{N}_{Tri}(y, y, y)$. Therefore, (4.26) and (4.27) imply that $\|\mathcal{N}y\|_{\mathcal{F}} \leq C\left(\|y\|^2_{\mathcal{Y}} + \|y\|^3_{\mathcal{Y}}\right)$, thus allowing us to conclude that $\Lambda : E \to G$ is well defined.
- Each term of $\Lambda : E \to G$ is linear except $\mathcal{N}y$. However, the bilinear and trilinear forms respectively given by (4.24) and (4.25) are continuous thanks to (4.26) and (4.27), and consequently, we deduce that $\Lambda \in C^1(E; G)$.

Let us take $\hat{e} = (0, 0)$, thus obtaining that $\Lambda(0, 0) = (0, 0)$. It follows that $\Lambda'(\hat{e}) : E \to G$ is given by $\Lambda'(\hat{e})(u, y) = (\mathcal{L}y - \mathbb{1}_\omega u, y(0, \cdot))$, whose surjectivity is equivalent to the null controllability of (1.8) that has been shown in Theorem 4.1. Therefore, we can apply Theorem 4.2 to get the desired result. The proof of Theorem 1.1 is complete. □

5 Further Remarks

In this section we give further remarks and address some natural questions that arise from the results obtained in this paper.

5.1 Other Boundary Conditions

In this paper we have addressed an internal control problem for the Cahn–Hilliard equation posed with boundary conditions (1.2), which actually are the relevant ones from the physical point of view according to [4, p. 259] together with [5, p. 160]. However, from the mathematical point of view, it is also interesting to consider other boundary conditions. For instance, instead of (1.2) we could have considered

$$z(t, 0) = z(t, L) = 0, \quad \partial_x^M z(t, 0) = \partial_x^M z(t, L) = 0, \quad t \in (0, T), \tag{5.1}$$

where $M \in \{1, 2\}$. If we replace (1.2) by (5.1), then Theorem 1.1 to remain valid requires Carleman estimate (3.7) to be valid in

$$\begin{aligned} \mathcal{Q} = \{q \in C^\infty([0, T] \times [0, L]) \,/\, q(\cdot, 0) = q(\cdot, L) = 0 \\ \text{and} \quad \partial_x^M q(\cdot, 0) = \partial_x^M q(\cdot, L) = 0\}, \end{aligned}$$

which can be checked to be the case thanks to the analysis required to deduce (3.35). Let us note that when $M = 1$ we recover the Carleman estimates given by [12, Theorem 3.3], [21, Proposition 2.1] and [33, Theorem 1.1] from the proof of Proposition 3.1. Therefore, Theorem 1.1 is still true if we replace (1.2) by (5.1).

5.2 Multi-dimensional Case

Let $\Omega \subset \mathbb{R}^N$, with $N \in \mathbb{N}$, be a non-empty bounded connected open set such that its boundary $\partial\Omega$ is of class C^2. Let $\omega \subset \Omega$ be a non-empty set. Let $n(x)$ be the outward unit normal vector to Ω at the point $x \in \partial\Omega$. As in the one-dimensional case addressed in this paper, the internal control problem to consider in the multi-dimensional case is the exact controllability to the trajectories of

$$\begin{cases} z_t + \Delta^2 z + \Delta z = \Delta\big(z^3\big) + u\mathbb{1}_\omega, & (t,x) \in (0,T)\times\Omega, \\ \dfrac{\partial z}{\partial n} = \dfrac{\partial \Delta z}{\partial n} = 0, & (t,x) \in (0,T)\times\partial\Omega, \\ z(0,x) = z_0(x), & x \in \Omega. \end{cases} \tag{5.2}$$

In order to prove Theorem 1.1 for the multi-dimensional case, we could follow the ideas used in this paper. If that is the case, then it would suffice a Carleman estimate for the operator $(-\partial_t + \Delta^2)$ defined on

$$\mathcal{Q} = \left\{ q \in C^\infty(\overline{[0,T]\times\Omega}) \;/\; \frac{\partial q}{\partial n} = \frac{\partial \Delta q}{\partial n} = 0, \quad (t,x) \in (0,T)\times\partial\Omega \right\}. \tag{5.3}$$

When doing the necessary changes to the proof of Proposition 3.1 in order to derive such a Carleman estimate, we will need to compute many terms, including

$$\begin{aligned} &\int_0^T \int_\Omega (-w_t)\mathcal{P}_2 w \, dxdt \\ &\quad = -\int_0^T \int_\Omega w_t \left[\nu^4 |\nabla\alpha|^4 w + 6\nu^2 |\nabla\alpha|^2 \Delta w + \Delta^2 w + 6\nu^2 \nabla(|\nabla\alpha|^2)\cdot\nabla w \right] dxdt. \end{aligned} \tag{5.4}$$

With the purpose to compare the one-dimensional case with the multi-dimensional one, let us consider the following boundary terms, which are two of the three that will appear when performing integrations by parts on Ω in (5.4).

$$B(w,\alpha) = -\int_0^T \int_{\partial\Omega} 6\nu^2 |\nabla\alpha|^2 w_t \frac{\partial w}{\partial n} \, d\sigma \, dt + \int_0^T \int_{\partial\Omega} \Delta w \frac{\partial w_t}{\partial n} \, d\sigma \, dt.$$

Assuming the existence of a weight function $\psi \in C^4(\overline{\Omega})$ such that $\psi > 0$ in Ω, $\psi = \Delta\psi = 0$ on $\partial\Omega$ and $|\nabla\psi| > 0$ in $\overline{\Omega \setminus \omega_0}$, where ω_0 is a non-empty open set satisfying that $\overline{\omega_0} \subset int(\omega)$, we see that the analysis required to deduce (3.35) allow us to obtain

$$B(w,\alpha) + B(\widetilde{w},\widetilde{\alpha}) = \int_0^T \int_{\partial\Omega} 4\nabla(e^{-\nu\alpha}) \cdot \nabla q \frac{\partial w_t}{\partial n} \, d\sigma \, dt. \tag{5.5}$$

Regarding the previous analysis, the following remarks are crucial.

(i) The existence of such a weight function $\psi \in C^4(\overline{\Omega})$ is not evident and is left as an open problem. Let us note that in the one-dimensional case we have asked it to satisfy (3.1)–(3.3) and we justified its existence by means of the example in Remark 3.1.

(ii) In the one-dimensional case the right-hand side of (5.5) vanishes, fact that is actually used in the deduction of (3.35). However, this is no longer the case in the multi-dimensional case.

The previous remarks tell us that further analysis is needed to derive the required Carleman estimate. Therefore, the validity of Theorem 1.1 in the multi-dimensional case is left as an open problem.

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