



Energy decay of a microbeam model with a locally distributed nonlinear feedback control



Patricio Guzmán

Departamento de Ingeniería Matemática y Centro de Modelamiento Matemático, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Santiago, Chile

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ABSTRACT

In this paper we address the problem of internal stabilization of the deflection of a microbeam, which is modeled by a sixth-order hyperbolic equation. Employing multiplier techniques and an integral inequality, we prove that a locally distributed nonlinear feedback control forces the energy associated to the deflection to decay exponentially or polynomially to zero. As a consequence of this, the deflection goes to the rest position as the time goes to infinity.

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1. Introduction

A microbeam is a beam whose dimensions are in the order of a few microns. According to [26, Chapter 6], the microbeams are perhaps the most common structural component used in micro-electro-mechanical systems (MEMS) such as actuators, filters, resonators and sensors.

The deflection $z = z(t, x)$ of a clamped microbeam of density $\rho > 0$, cross-sectional area $A > 0$, Young's modulus $E > 0$, area moment of inertia $I > 0$, shear modulus $G > 0$ and length $L > 0$ being subjected to a distributed load $f = f(t, x)$ can be modeled by

$$\left\{ \begin{array}{l} \rho A z_{tt} + M_1 z_{xxxx} - M_2 z_{xxxxxx} = f, \quad (t, x) \in (0, \infty) \times (0, L), \\ z(t, 0) = z_x(t, 0) = z_{xx}(t, 0) = 0, \quad t \in (0, \infty), \\ z(t, L) = z_x(t, L) = z_{xx}(t, L) = 0, \quad t \in (0, \infty), \\ z(0, x) = z_0(x), \quad x \in (0, L), \\ z_t(0, x) = z_1(x), \quad x \in (0, L). \end{array} \right. \quad (1.1)$$

E-mail address: pguzman@dim.uchile.cl.

This model has been derived in [12, Section 3] and [17, Section 4] by using the modified strain gradient elasticity theory developed in [19, Section 2] together with Hamilton’s Principle. Here

$$M_1 = EI + GA \left(2l_0^2 + \frac{8}{15}l_1^2 + l_2^2 \right) \quad \text{and} \quad M_2 = GA \left(2l_0^2 + \frac{4}{5}l_1^2 \right),$$

where $l_0 > 0$, $l_1 > 0$ and $l_2 > 0$ are the material length scale parameters associated with dilatation gradients, deviatoric stretch gradients and rotation gradients, respectively. These parameters were introduced in [19, Section 2] to characterize the phenomenon observed in experiments that the deformation of some materials is size-dependent. Further information on microbeams, its related experiments and also on MEMS may be found in [12,17,19,26] and the references therein.

Without loss of generality, and in order to ease the notation throughout this paper, from now on instead of considering (1.1) we consider

$$\left\{ \begin{array}{l} z_{tt} + z_{xxxx} - z_{xxxxx} = f, \quad (t, x) \in (0, \infty) \times (0, 1), \\ z(t, 0) = z_x(t, 0) = z_{xx}(t, 0) = 0, \quad t \in (0, \infty), \\ z(t, 1) = z_x(t, 1) = z_{xx}(t, 1) = 0, \quad t \in (0, \infty), \\ z(0, x) = z_0(x), \quad x \in (0, 1), \\ z_t(0, x) = z_1(x), \quad x \in (0, 1). \end{array} \right. \tag{1.2}$$

In this paper we address the problem of internal stabilization of (1.2). Let us present such a problem. For a regular enough solution $z = z(t, x)$ of (1.2) we define its energy by

$$E(t) = \frac{1}{2} \int_0^1 (|z_t|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx, \quad t \in [0, \infty). \tag{1.3}$$

Then, formal computations yield

$$E'(t) = \int_0^1 f z_t dx, \quad t \in [0, \infty), \tag{1.4}$$

from which it follows that the energy is conserved when $f = f(t, x)$ is not present. With the purpose of stabilizing $z = z(t, x)$ to the rest position, that is to say, $\lim_{t \rightarrow \infty} z(t, \cdot) = 0$ and $\lim_{t \rightarrow \infty} z_t(t, \cdot) = 0$, we take a $f = f(t, x)$ that forces the dissipation of the energy, and moreover, its decay.

From (1.4) we see that we could choose

$$\left. \begin{array}{l} f(t, x) = f(z_t(t, x)) = -a(x)z_t(t, x) \quad \text{with} \\ a \in L^\infty(0, 1) \quad \text{such that} \quad a(x) \geq a_0 > 0 \quad \text{for almost every} \quad x \in (0, 1). \end{array} \right\} \tag{1.5}$$

With the choice of (1.5), which is a linear feedback control distributed over the whole domain, the dissipation of the energy is ensured. Moreover, the exponential decay of the energy can be obtained by applying the ideas introduced by Haraux and Zuazua in [11] to

$$E_\varepsilon(t) = E(t) + \varepsilon \int_0^1 z z_t dx, \quad t \in [0, \infty), \tag{1.6}$$

where $\varepsilon > 0$ is a constant that must be suitably chosen. Let us note that (1.6) can be seen as a perturbation of (1.3). Those ideas were also applied in [3,6,7,16,21,28,29] to obtain the exponential or polynomial decay

of the energy associated to the solution of some hyperbolic equations with linear or nonlinear feedback controls acting either in the domain or at the boundary.

Even if (1.5) solves our stabilization problem, acting over the whole domain might be too restrictive. For this reason instead of choosing (1.5) we choose a nonlinear feedback control locally distributed over the domain, for being less restrictive and more general. In order to present it, we need to make some assumptions.

(A1) (Localization) Let $\omega \subset (0, 1)$ be a non-empty interval. Let $a \in L^\infty(0, 1)$ be a non-negative function such that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$.

(A2) (Nonlinearity) Let $g \in C(\mathbb{R})$ be a non-decreasing function satisfying that there exist constants $R \in [1, \infty)$, $S \in [1, \infty)$ and $(n_1, n_2, n_3, n_4) \in (0, \infty)^4$ such that $n_1|\tau|^R \leq |g(\tau)| \leq n_2|\tau|^{\frac{1}{R}}$ if $|\tau| \leq 1$ and $n_3|\tau| \leq |g(\tau)| \leq n_4|\tau|^S$ if $|\tau| > 1$.

Then, with the previous assumptions in mind we choose

$$f(t, x) = f(z_t(t, x)) = -a(x)g(z_t(t, x)). \tag{1.7}$$

Remark 1.1. In (A1) we could take $a(x) = \mathbb{1}_\omega(x)$, where $\mathbb{1}_\omega$ denotes the characteristic function on ω , that is to say, $\mathbb{1}_\omega(x) = 1$ if $x \in \omega$ and $\mathbb{1}_\omega(x) = 0$ if $x \notin \omega$. The assumption that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$ in (A1) means that (1.7) acts only through ω .

Remark 1.2. In (A2) we could take $g(\tau) = |\tau|^{R-1}\tau$ for any $R \in [1, \infty)$. The type of nonlinearity in (A2) is similar to the ones in [13,22], the same as the ones in [8,15,18] and more general than the ones in [3,9,21, 28,29]. Finally, from (A2) we deduce

$$(g(\tau_1) - g(\tau_2))(\tau_1 - \tau_2) \geq 0 \quad \text{for every } (\tau_1, \tau_2) \in \mathbb{R}^2 \quad \text{and } g(0) = 0. \tag{1.8}$$

In virtue of (1.4) and (1.7) we have

$$E'(t) = - \int_0^1 ag(z_t)z_t dx, \quad t \in [0, \infty), \tag{1.9}$$

from which it follows that the energy still dissipates due to (A1) and (1.8). Therefore, it is natural to wonder if the energy still decays. Our main result gives a positive answer to this question, thus also giving an answer to our stabilization problem.

Theorem 1.1. *Let us assume (A1) and (A2). Let $(z_0, z_1) \in H_0^3(0, 1) \times L^2(0, 1)$. Then, (1.2) with (1.7) has a unique weak solution $z \in C([0, \infty); H_0^3(0, 1)) \cap C^1([0, \infty); L^2(0, 1))$. Moreover, its energy decays exponentially or polynomially to zero. Being precise:*

- (Exponential decay) If $R = 1$, then there exists a constant $C_1 > 0$ such that

$$E(t) \leq E(0) \exp \left\{ 1 - \frac{t}{C_1 \left(1 + E(0)^{\frac{S-1}{2S}} \right)} \right\}, \quad t \in [0, \infty). \tag{1.10}$$

- (Polynomial decay) If $R > 1$, then there exists a constant $C_2 > 0$ such that

$$E(t) \leq C_2 \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right)^{\frac{2}{R-1}} t^{-\frac{2}{R-1}}, \quad t \in (0, \infty). \tag{1.11}$$

Let us make some comments on the proof of this theorem. Its well-posedness part is shown with the aid of the semigroup theory, while its energy decay part is shown by adapting a method introduced by Tebou in [22] to address the problem of internal stabilization of the wave equation. We have followed this approach because we have not been able to find a perturbation of (1.3), like the one given in (1.6), to prove (1.10) and (1.11).

The method introduced by Tebou in [22] allows to obtain, by employing multiplier techniques and an integral inequality derived by Komornik in [13], the exponential or polynomial decay of the energy associated to the solution of a large class of hyperbolic equations with locally distributed nonlinear feedback controls. However, that method to work requires the nonlinear feedback control to be localized on a suitable neighborhood of the boundary. Then, we see from [22, Pages 502 and 503] that we should have assumed (A0) instead of (A1), where:

(A0) (Localization) Let $\varepsilon \in (0, 1)$. Let us consider either $\omega = (0, \varepsilon)$ or $\omega = (1 - \varepsilon, 1)$. Let $a \in L^\infty(0, 1)$ be a non-negative function such that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$.

One of the contributions of this paper is the adaptation of the method introduced by Tebou in [22] to the one-dimensional case in such a way that (1.10) and (1.11) can still be shown with a weaker assumption than (A0), namely (A1).

Regarding the study of the control properties of microbeams, we mention that the exact boundary controllability was analyzed in [10,24,27], the boundary stabilization with linear feedback controls was addressed in [23,25] and the exact boundary observability was studied in [5].

The rest of this paper is organized as follows. In Section 2 we prove the well-posedness results needed for the study of (1.2) with (1.7). Then, in Section 3 we prove our main result, which is the exponential or polynomial decay of the energy given by Theorem 1.1.

2. Well-posedness

In this section we apply the semigroup theory exposed in [1] to obtain the well-posedness results needed for the study of (1.2) with (1.7). We proceed as in [8,9,14]. Let us introduce some notations. Let us consider the Hilbert spaces $H = L^2(0, 1)$ and $V = H_0^3(0, 1)$ respectively equipped with the inner products

$$(z, y)_H = \int_0^1 zy \, dx \quad \text{and} \quad (z, y)_V = \int_0^1 (z''y'' + z'''y''') \, dx.$$

Note that the latter can be done due to Poincaré’s inequality. We identify H with its dual space H^* to obtain the continuous and dense injections $V \hookrightarrow H \hookrightarrow V^*$, where $V^* = H^{-3}(0, 1)$ is the dual space of V . We employ $\langle \cdot, \cdot \rangle_{V^* \times V}$ to denote the duality product between V^* and V . Let us define the operators $A : V \rightarrow V^*$ and $B : V \rightarrow H$ respectively through

$$\langle Az, y \rangle_{V^* \times V} = (z, y)_V \quad \text{and} \quad Bz = ag(z). \tag{2.1}$$

Lemma 2.1. *The operators defined through (2.1) are well defined.*

Proof. The operator $A : V \rightarrow V^*$ is defined through a bilinear continuous form, thus it is well defined. Introducing the sets $\Omega_1 = \{x \in (0, 1) / |z(x)| \leq 1\}$ and $\Omega_2 = \{x \in (0, 1) / |z(x)| > 1\}$ we see that the

operator $B : V \rightarrow H$ is also well defined. Indeed, thanks to (A2), Hölder’s inequality and the continuous injection $V \hookrightarrow L^\infty(0, 1)$ there exists a constant $C > 0$ such that

$$\begin{aligned} \|Bz\|_H^2 &\leq \|a\|_{L^\infty(\Omega_1)}^2 n_2^2 \int_{\Omega_1} |z|^{\frac{2}{R}} dx + \|a\|_{L^\infty(\Omega_2)}^2 n_4^2 \|z\|_{L^\infty(\Omega_2)}^{2S-2} \int_{\Omega_2} |z|^2 dx \\ &\leq C \left(\int_{\Omega_1} |z|^2 dx \right)^{\frac{1}{R}} + C \|z\|_V^{2S-2} \int_{\Omega_2} |z|^2 dx, \quad z \in V. \end{aligned} \tag{2.2}$$

The proof of Lemma 2.1 is complete. \square

Additionally, we also consider the Hilbert space $\mathcal{H} = V \times H$ equipped with the inner product

$$((z, y), (Z, Y))_{\mathcal{H}} = (z, Z)_V + (y, Y)_H.$$

Finally, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ by

$$D(\mathcal{A}) = \{(z, y) \in V \times V / Az + By \in H\} \quad \text{and} \quad \mathcal{A}(z, y) = (-y, Az + By). \tag{2.3}$$

Lemma 2.2. *We have that $D(\mathcal{A}) = (H^6(0, 1) \cap V) \times V$.*

Proof. Let us show that $(H^6(0, 1) \cap V) \times V \subset D(\mathcal{A})$. By Lemma 2.1 we only need to show that $z \in H^6(0, 1) \cap V$ implies that $Az \in H$. Assuming that $z \in H^6(0, 1) \cap V$, three integrations by parts and the fact that $z'''' - z'''''' \in H$ allow us to deduce the desired result since

$$\langle Az, \phi \rangle_{V^* \times V} = (z, \phi)_V = (z'''' - z''''', \phi)_H = (z'''' - z''''', \phi)_{V^* \times V}, \quad \phi \in V.$$

Now, let us show that $D(\mathcal{A}) \subset (H^6(0, 1) \cap V) \times V$. By Lemma 2.1 we only need to show that $z \in V$ and $Az \in H$ imply that $z \in H^6(0, 1) \cap V$. Assuming that $z \in V$ and $Az \in H$, one integration by parts yields

$$(Az, \phi)_H = \langle Az, \phi \rangle_{V^* \times V} = (z, \phi)_V = (-1)^3 (z' - z''', \phi''')_H, \quad \phi \in C_0^\infty(0, 1),$$

from which it follows that $Az = (z' - z''')''''$ in the sense of distributions. Finally, taking into account [2, Chapter 8, Remark 7], we deduce the desired result since $z' - z''' \in H$ and $(z' - z''')'''' \in H$. The proof of Lemma 2.2 is complete. \square

Let us note that in the proof of Lemma 2.2 we have deduced that $Az = z'''' - z''''''$ if $z \in H^6(0, 1) \cap V$. Then, with the aid of the operator defined by (2.3) and Lemma 2.2 we see that (1.2) with (1.7) can be written as

$$\left. \begin{aligned} \frac{d}{dt} (z, z_t) + \mathcal{A}(z, z_t) &= 0, \quad t \in [0, \infty), \\ (z, z_t)(0) &= (z_0, z_1). \end{aligned} \right\} \tag{2.4}$$

We are going to obtain some well-posedness results for (1.2) with (1.7) by applying the semigroup theory to (2.4). The main result of this section is the following one.

Proposition 2.1. *The operator defined by (2.3) is a maximal monotone operator in \mathcal{H} .*

Proof. In view of [1, Proposition 2.2] we see that it suffices to prove that the operator \mathcal{A} is monotone and that $R(I + \mathcal{A}) = \mathcal{H}$. Let us prove the first part. Let (z, y) and (Z, Y) be elements of $D(\mathcal{A})$. Then, thanks to Lemma 2.2, three integrations by parts, (A1) and (1.8) we have

$$\begin{aligned} & (\mathcal{A}(z, y) - \mathcal{A}(Z, Y), (z, y) - (Z, Y))_{\mathcal{H}} \\ &= ((-(y - Y), A(z - Z)), (z - Z, y - Y))_{\mathcal{H}} + ((0, By - BY), (z - Z, y - Y))_{\mathcal{H}} \\ &= -(y - Y, z - Z)_V + (A(z - Z), y - Y)_H + (By - BY, y - Y)_H \\ &= (By - BY, y - Y)_H \geq 0. \end{aligned}$$

Now, let us prove the second part, that is to say, to prove that for any $(z_0, y_0) \in \mathcal{H}$ there exists $(z, y) \in D(\mathcal{A})$ such that $(I + \mathcal{A})(z, y) = (z - y, y + Az + By) = (z_0, y_0)$. To this end, for $y_0 - Az_0 \in V^*$ let us consider the problem:

$$\text{Find } y \in V \text{ such that } y + Ay + By = y_0 - Az_0 \text{ in } V^*. \tag{2.5}$$

Let us assume that we have solved (2.5). Then, introducing $z = y + z_0 \in V$ we see from (2.5) that $Az + By = y_0 - y$ in V^* , which tells us that $Az + By \in H$ because $y_0 - y \in H$. Accordingly, we have found a $(z, y) \in D(\mathcal{A})$ such that $(I + \mathcal{A})(z, y) = (z - y, y + Az + By) = (z_0, y_0)$. Now, it only left to solve (2.5). In order to do so, let us define the functional $J : V \rightarrow \mathbb{R}$ by

$$J(y) = \frac{1}{2}\|y\|_H^2 + \frac{1}{2}\|y\|_V^2 + \int_0^1 G(y) dx - \langle y_0 - Az_0, y \rangle_{V^* \times V} \quad \text{with} \quad G(s) = a \int_0^s g(t) dt.$$

Since (A1) and (1.8) imply

$$0 \leq G(y) \leq (By)y, \quad y \in V, \tag{2.6}$$

it follows that the functional is well defined due to the Cauchy–Schwarz inequality and (2.2). Indeed, there exists a constant $C > 0$ such that

$$|J(y)| \leq \frac{1}{2}\|y\|_H^2 + \frac{1}{2}\|y\|_V^2 + C \left(\|y\|_H^{\frac{1}{\alpha}} + \|y\|_V^{S-1} \|y\|_H \right) \|y\|_H + \|y_0 - Az_0\|_{V^*} \|y\|_V, \quad y \in V.$$

Now, let us show that the functional is convex, coercive and lower-semicontinuous. In view of (A1) and (1.8) we get

$$(G'(y_1) - G'(y_2))(y_1 - y_2) \geq 0, \quad (y_1, y_2) \in V \times V,$$

which allows us to conclude that the functional is convex (see [20, Proposition 3.10]). From (2.6) and the Cauchy–Schwarz inequality we get

$$J(y) \geq \frac{1}{2}\|y\|_V^2 - \|y_0 - Az_0\|_{V^*} \|y\|_V, \quad y \in V,$$

thus implying that the functional is coercive because $J(y) \rightarrow \infty$ as $\|y\|_V \rightarrow \infty$. Finally, the functional is lower-semicontinuous because it is continuous. Indeed, by (1.8), (2.2) and the Cauchy–Schwarz inequality there exists a constant $C > 0$ such that

$$\begin{aligned} \|G(y_1) - G(y_2)\|_{L^1(0,1)} &\leq \|By_2\|_H \|y_1 - y_2\|_H \\ &\leq C \left(\|y_2\|_H^{\frac{1}{2}} + \|y_2\|_V^{S-1} \|y_2\|_H \right) \|y_1 - y_2\|_H, \quad (y_1, y_2) \in V \times V. \end{aligned}$$

Therefore, we infer the existence of a $\hat{y} \in V$ minimizing the functional (see [20, Theorem 2.19]). Since such a minimizer $\hat{y} \in V$ satisfies

$$0 = \langle J'(\hat{y}), \phi \rangle_{V^* \times V} = \langle \hat{y} + A\hat{y} + B\hat{y} - (y_0 - Az_0), \phi \rangle_{V^* \times V}, \quad \phi \in V,$$

we see that $\hat{y} \in V$ solves (2.5). The proof of Proposition 2.1 is complete. \square

Let us apply the semigroup theory to (2.4). For every $(z_0, z_1) \in D(\mathcal{A})$ we have that (2.4) has a unique strong solution $(z, z_t) \in L^\infty(0, \infty; D(\mathcal{A})) \cap W^{1,\infty}(0, \infty, \mathcal{H})$ in virtue of Proposition 2.1 and [1, Theorem 3.1]. Furthermore, its solution map $(z_0, z_1) \in D(\mathcal{A}) \subset \mathcal{H} \mapsto (z, z_t)(t, \cdot) \in \mathcal{H}$ is a contraction for each $t \in [0, \infty)$. Since $D(\mathcal{A})$ is dense in \mathcal{H} due to Lemma 2.2, such a solution map can be extended by continuity to a unique map $S(t) : \mathcal{H} \rightarrow \mathcal{H}$ so that $(z, z_t)(t, \cdot) = S(t)(z_0, z_1)$ for each $t \in [0, \infty)$ when $(z_0, z_1) \in D(\mathcal{A})$. It can be checked that $(S(t))_{t \geq 0}$ is a strongly continuous semigroup of contractions in \mathcal{H} (see [1, Page 55]), thus for any $(z_0, z_1) \in \mathcal{H}$ we may define the unique weak solution of (2.4) by $S(t)(z_0, z_1)$ with the property that $t \in [0, \infty) \mapsto S(t)(z_0, z_1) \in \mathcal{H}$ is a continuous map.

Remark 2.1. For a given $(z_0, z_1) \in \mathcal{H}$ let us take the unique weak solution $S(t)(z_0, z_1)$ of (2.4). Since $D(\mathcal{A})$ is dense in \mathcal{H} due to Lemma 2.2, there exists a sequence $\{(z_{0n}, z_{1n})\}_{n \in \mathbb{N}} \subset D(\mathcal{A})$ such that $(z_{0n}, z_{1n}) \rightarrow (z_0, z_1)$ in \mathcal{H} as $n \rightarrow \infty$. Now, for each $n \in \mathbb{N}$ let us take the corresponding unique strong solution $S(t)(z_{0n}, z_{1n})$ of (2.4). Therefore, since $(S(t))_{t \geq 0}$ is a strongly continuous semigroup of contractions in \mathcal{H} , for each $t \in [0, \infty)$ we have that $S(t)(z_{0n}, z_{1n}) \rightarrow S(t)(z_0, z_1)$ in \mathcal{H} as $n \rightarrow \infty$.

The previous analysis is translated into the following result.

Proposition 2.2. *Let us assume (A1) and (A2).*

- (i) *Let $(z_0, z_1) \in (H^6(0,1) \cap V) \times V$. Then, (1.2) with (1.7) has a unique strong solution $z \in L^\infty(0, \infty; H^6(0,1) \cap V) \cap W^{1,\infty}(0, \infty; V) \cap W^{2,\infty}(0, \infty; H)$.*
- (ii) *Let $(z_0, z_1) \in V \times H$. Then, (1.2) with (1.7) has a unique weak solution $z \in C([0, \infty); V) \cap C^1([0, \infty); H)$.*

3. Internal stabilization

In this section we prove our main result, which is the exponential or polynomial decay of the energy given by Theorem 1.1. To this end, we are going to apply an integral inequality derived by Komornik that can be found in [13, Theorem 1.4] or [14, Theorem 9.1].

Theorem 3.1. *Let $E : [0, \infty) \rightarrow [0, \infty)$ be a non-increasing function. Assume that there exist constants $\alpha \geq 0$ and $\beta > 0$ such that*

$$\int_t^\infty E(\tau)^{\alpha+1} d\tau \leq \beta E(t), \quad t \in [0, \infty). \tag{3.1}$$

Then, we have

$$E(t) \leq \begin{cases} E(0) \exp \left\{ 1 - \frac{t}{\beta} \right\}, & t \in [0, \infty), \quad \text{if } \alpha = 0, \\ \beta^{\frac{1}{\alpha}} \left(1 + \frac{1}{\alpha} \right)^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}}, & t \in (0, \infty), \quad \text{if } \alpha > 0. \end{cases} \tag{3.2}$$

Remark 3.1. The second line of the right-hand side of (3.2), corresponding to the case $\alpha > 0$, is a consequence of [13, Section 1, (1.22)] or [14, Section 9, (2)]. We have preferred the current expression to clearly appreciate the polynomial behavior.

Proof of Theorem 1.1. In virtue of the density argument presented in Remark 2.1, it suffices to perform all the required computations assuming that $(z_0, z_1) \in (H^6(0, 1) \cap H_0^3(0, 1)) \times H_0^3(0, 1)$, so that (1.2) with (1.7) has by Proposition 2.2 a unique strong solution $z \in L^\infty(0, \infty; H^6(0, 1) \cap H_0^3(0, 1)) \cap W^{1,\infty}(0, \infty; H_0^3(0, 1)) \cap W^{2,\infty}(0, \infty; L^2(0, 1))$.

In view of (A1) and (1.8) we see that multiplying (1.2) with (1.7) by $z_t(t, x)$ and then performing some integrations by parts on $(0, 1)$ yield

$$E'(t) = - \int_0^1 ag(z_t)z_t \, dx \leq 0, \quad t \in [0, \infty), \tag{3.3}$$

which tells us that the energy is a non-increasing function. Then, in order to apply Theorem 3.1 we need to obtain constants $\alpha \geq 0$ and $\beta > 0$ such that the energy satisfies (3.1). This will be done by employing several multiplier techniques. We split the proof into four parts.

Part 1. Let $0 \leq t_1 < t_2 < \infty$. Let $m \in C^\infty([0, 1])$ be such that $m(0) = 0$. Let $\alpha \geq 0$ to be determined later. Multiplying (1.2) with (1.7) by $E(t)^\alpha m(x)z_x(t, x)$ and then performing some integrations by parts on $(t_1, t_2) \times (0, 1)$ we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^1 \frac{1}{2} E^\alpha m' (|z_t|^2 + 3|z_{xx}|^2 + 5|z_{xxx}|^2) dx dt + \int_0^1 E^\alpha m z_x z_t \Big|_{t_1}^{t_2} dx \\ & - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' m z_x z_t \, dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha m'' (z_{xx} z_x + 2z_{xxx} z_{xx}) dx dt \\ & + \int_{t_1}^{t_2} \int_0^1 E^\alpha (m''' z_x + m'' z_{xx}) z_{xxx} dx dt \\ & + \int_{t_1}^{t_2} \int_0^1 E^\alpha m z_x ag(z_t) \, dx dt = \int_{t_1}^{t_2} \frac{1}{2} E^\alpha m(1) |z_{xxx}(t, 1)|^2 dt. \end{aligned} \tag{3.4}$$

Then, choosing $m(x) = x$ in (3.4) leads us to

$$\begin{aligned} & \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt + \int_0^1 E^\alpha x z_x z_t \Big|_{t_1}^{t_2} dx \\ & - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' x z_x z_t \, dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha x z_x ag(z_t) \, dx dt = \int_{t_1}^{t_2} \frac{1}{2} E^\alpha |z_{xxx}(t, 1)|^2 dt. \end{aligned} \tag{3.5}$$

For a fixed $x_0 \in \text{int}(\omega)$ let us introduce $\varepsilon_0 = \text{dist}(x_0, \partial\omega)/2 > 0$ and $\omega_0 = (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$, thus obtaining that $\overline{\omega_0} \subset \omega$. Then, let us take a function $\chi_0 \in C^\infty([0, 1])$ satisfying that $\chi_0 = 0$ in $[0, x_0 - \varepsilon_0]$ and $\chi_0 = 1$ in $[x_0 + \varepsilon_0, 1]$. This time we see that choosing $m(x) = \chi_0(x)$ in (3.4) and then using Cauchy’s inequality we get a constant $C_1 > 0$ such that

$$\begin{aligned} \int_{t_1}^{t_2} \frac{1}{2} E^\alpha |z_{xxx}(t, 1)|^2 dt &\leq \int_0^1 E^\alpha \chi_0 z_x z_t \Big|_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_0 z_x z_t dx dt \\ &+ \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_0 z_x ag(z_t) dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt \\ &+ C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt, \end{aligned}$$

and hence, from (3.5) it follows

$$\begin{aligned} \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt &\leq \int_0^1 E^\alpha (-x + \chi_0) z_x z_t \Big|_{t_1}^{t_2} dx \\ &+ \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' (x - \chi_0) z_x z_t dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (-x + \chi_0) z_x ag(z_t) dx dt \\ &+ C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt. \end{aligned} \tag{3.6}$$

Let us handle the first two terms of the right-hand side of (3.6). Using that (3.3) implies that $E(t_2) \leq E(t_1) \leq E(0)$ whenever $0 \leq t_1 < t_2 < \infty$, the Cauchy and Poincaré inequalities give us a constant $C_2 > 0$ such that

$$\int_0^1 E^\alpha (-x + \chi_0) z_x z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' (x - \chi_0) z_x z_t dx dt \leq C_2 E(0)^\alpha E(t_1). \tag{3.7}$$

To end this part, we see that plugging (3.7) into (3.6) we arrive at

$$\begin{aligned} \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z_{xx}|^2 + 2|z_{xxx}|^2) dx dt &\leq C_2 E(0)^\alpha E(t_1) \\ &+ \int_{t_1}^{t_2} \int_0^1 E^\alpha (-x + \chi_0) z_x ag(z_t) dx dt + C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha |z_t|^2 dx dt \\ &+ C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_x|^2 + |z_{xx}|^2 + |z_{xxx}|^2) dx dt. \end{aligned} \tag{3.8}$$

Part 2. In this part we handle the right-hand side of (3.8). Let us take a non-negative function $\chi_1 \in C_0^\infty(\omega)$ such that $\chi_1 = C_1$ in ω_0 . Note that this is possible because $\overline{\omega_0} \subset \omega$. Multiplying (1.2) with (1.7) by $E(t)^\alpha \chi_1(x)z(t, x)$ and then performing some integrations by parts on $(t_1, t_2) \times (0, 1)$ we obtain

$$\begin{aligned}
 C_1 \int_{t_1}^{t_2} \int_{\omega_0} E^\alpha (|z_{xx}|^2 + |z_{xxx}|^2) dx dt &\leq - \int_0^1 E^\alpha \chi_1 z z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_1 z z_t dx dt \\
 &- \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1'' z + 2\chi_1' z_x) z_{xx} dx dt - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1''' z + 3\chi_1'' z_x + 3\chi_1' z_{xx}) z_{xxx} dx dt \\
 &- \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_1 z ag(z_t) dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_1 |z_t|^2 dx dt.
 \end{aligned} \tag{3.9}$$

To handle the first four terms of the right-hand side of (3.9) we can proceed as we did for the obtention of (3.7) to get a constant $C_3 > 0$ such that

$$\begin{aligned}
 &- \int_0^1 E^\alpha \chi_1 z z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi_1 z z_t dx dt - \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1'' z + 2\chi_1' z_x) z_{xx} dx dt \\
 &- \int_{t_1}^{t_2} \int_0^1 E^\alpha (\chi_1''' z + 3\chi_1'' z_x + 3\chi_1' z_{xx}) z_{xxx} dx dt \leq C_3 E(0)^\alpha E(t_1) \\
 &+ \frac{1}{2} \int_{t_1}^{t_2} \int_0^1 E^\alpha |z_{xxx}|^2 dx dt + C_3 \int_{t_1}^{t_2} \int_\omega E^\alpha (|z|^2 + |z_x|^2 + |z_{xx}|^2) dx dt.
 \end{aligned} \tag{3.10}$$

Accordingly, plugging (3.10) into (3.9), and then, the resulting expression into (3.8), we find a constant $C_4 > 0$ such that

$$\begin{aligned}
 &\int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(|z_{xx}|^2 + \frac{3}{2} |z_{xxx}|^2 \right) dx dt \leq C_4 E(0)^\alpha E(t_1) \\
 &+ C_4 \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z| + |z_x|) |ag(z_t)| dx dt + C_4 \int_{t_1}^{t_2} \int_\omega E^\alpha |z_t|^2 dx dt \\
 &+ C_4 \int_{t_1}^{t_2} \int_\omega E^\alpha (|z|^2 + |z_x|^2 + |z_{xx}|^2) dx dt.
 \end{aligned} \tag{3.11}$$

In order to handle the last term of the right-hand side of (3.11) we recall useful convexity inequalities for Sobolev spaces that can be found in [4, Chapter IV, (7.37)]; namely, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}
 \int_\omega |z_x|^2 dx &\leq \frac{1}{\varepsilon_1} \int_\omega |z_{xx}|^2 dx + C_5 \varepsilon_1 \int_\omega |z|^2 dx, \quad \varepsilon_1 > 0, \\
 \int_\omega |z_{xx}|^2 dx &\leq \frac{1}{\varepsilon_2} \int_\omega |z_{xxx}|^2 dx + C_5 \varepsilon_2^2 \int_\omega |z|^2 dx, \quad \varepsilon_2 > 0.
 \end{aligned}$$

Considering $\varepsilon_1 = 2C_4 > 0$ and $\varepsilon_2 = C_4 > 0$, we apply the previous two convexity inequalities for Sobolev spaces in (3.11) and then take into account that $a(x) \geq a_0 > 0$ for almost every $x \in \omega$ to get a constant $C_6 > 0$ such that

$$\begin{aligned} & \int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2}|z_{xx}|^2 + \frac{1}{2}|z_{xxx}|^2 \right) dx dt \leq C_4 E(0)^\alpha E(t_1) \\ & + C_4 \int_{t_1}^{t_2} \int_0^1 E^\alpha (|z| + |z_x|) |ag(z_t)| dx dt \\ & + C_4 \int_{t_1}^{t_2} \int_0^1 E^\alpha a |z_t|^2 dx dt + C_6 \int_{t_1}^{t_2} \int_0^1 E^\alpha a |z|^2 dx dt. \end{aligned} \tag{3.12}$$

Part 3. In this part we handle the last term of the right-hand side of (3.12). To this end, we adapt an idea introduced by Conrad and Rao in [3] that consists in selecting a multiplier as the weak solution of a suitable elliptic problem. Let $\Omega = (0, 1)$. As in [22, Page 515], for each $t \in [t_1, t_2]$ let us consider

$$\begin{cases} \chi_{xxxx} - \chi_{xxxxx} = az & \text{in } \Omega, \\ \chi = \chi_x = \chi_{xx} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

On the one hand, since $a(\cdot)z(t, \cdot) \in L^2(0, 1)$ for each $t \in [t_1, t_2]$, the Lax–Milgram theorem tells us that (3.13) has a unique weak solution $\chi(t, \cdot) \in H_0^3(0, 1)$ satisfying

$$\int_0^1 |\chi_{xx}(t, x)|^2 dx + \int_0^1 |\chi_{xxx}(t, x)|^2 dx = \int_0^1 a(x)z(t, x)\chi(t, x) dx, \quad t \in [t_1, t_2].$$

Here we see that the Cauchy and Poincaré inequalities give us a constant $C_7 > 0$ such that

$$\int_0^1 |\chi(t, x)|^2 dx + \int_0^1 |\chi_x(t, x)|^2 dx \leq C_7 \int_0^1 |a(x)z(t, x)|^2 dx, \quad t \in [t_1, t_2]. \tag{3.14}$$

On the other hand, since $a(\cdot)z_t(t, \cdot) \in L^2(0, 1)$ for each $t \in [t_1, t_2]$, we can proceed as we just did for the obtention of (3.14) to conclude that $\chi_t(t, \cdot) \in H_0^3(0, L)$ satisfies

$$\int_0^1 |\chi_t(t, x)|^2 dx \leq C_7 \int_0^1 |a(x)z_t(t, x)|^2 dx, \quad t \in [t_1, t_2]. \tag{3.15}$$

Multiplying (1.2) with (1.7) by $E(t)^\alpha \chi(t, x)$, then performing some integrations by parts on $(t_1, t_2) \times (0, 1)$ and finally recalling that $\chi(t, x)$ is the unique weak solution of (3.13) we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \int_0^1 E^\alpha a |z|^2 dx dt = - \int_0^1 E^\alpha \chi z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi z_t dx dt \\ & - \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi ag(z_t) dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \chi_t z_t dx dt. \end{aligned} \tag{3.16}$$

To handle the first two terms of the right-hand side of (3.16), we consider (3.14) and then proceed as we did for the obtention of (3.7) to get a constant $C_8 > 0$ such that

$$-\int_0^1 E^\alpha \chi z_t \Big|_{t_1}^{t_2} dx + \int_{t_1}^{t_2} \int_0^1 \alpha E^{\alpha-1} E' \chi z_t dx dt \leq C_8 E(0)^\alpha E(t_1).$$

To end this part, we see that plugging the previous inequality into (3.16), and then, the resulting expression into (3.12), we find a constant $C_9 > 0$ such that

$$\begin{aligned} &\int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt \leq C_9 E(0)^\alpha E(t_1) \\ &+ C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha (|\chi| + |z| + |z_x|) |ag(z_t)| dx dt \\ &+ C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha a |z_t|^2 dx dt + C_9 \int_{t_1}^{t_2} \int_0^1 E^\alpha |\chi_t z_t| dx dt. \end{aligned} \tag{3.17}$$

Part 4. In this part we handle the last three terms of the right-hand side of (3.17). In this part we consider a constant $C_{10} > 0$ that might vary from line to line. On the one hand, the continuous injection $H^1(0, 1) \hookrightarrow L^\infty(0, 1)$, (3.14) and Poincaré’s inequality allow us to obtain

$$\int_{t_1}^{t_2} \int_0^1 E^\alpha (|z| + |z_x| + |\chi|) |ag(z_t)| dx dt \leq C_{10} \int_{t_1}^{t_2} \int_0^1 E^\alpha E^{\frac{1}{2}} |ag(z_t)| dx dt.$$

On the other hand, the Cauchy–Schwarz inequality and (3.15) imply

$$\int_{t_1}^{t_2} \int_0^1 E^\alpha a |z_t|^2 dx dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha |\chi_t z_t| dx dt \leq C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} \left(\int_0^1 |az_t|^2 dx \right)^{\frac{1}{2}} dt.$$

Accordingly, combining (3.17) with the previous two inequalities lead us to

$$\begin{aligned} &\int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt \leq C_{10} E(0)^\alpha E(t_1) \\ &+ C_{10} \int_{t_1}^{t_2} \int_0^1 E^\alpha E^{\frac{1}{2}} |ag(z_t)| dx dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} \left(\int_0^1 |az_t|^2 dx \right)^{\frac{1}{2}} dt. \end{aligned} \tag{3.18}$$

In order to handle the last two terms of the right-hand side of (3.18), for each $t \in [t_1, t_2]$ we introduce the sets $\Omega_1(t) = \{x \in (0, 1) / |z_t(t, x)| \leq 1\}$ and $\Omega_2(t) = \{x \in (0, 1) / |z_t(t, x)| > 1\}$. Setting $p_1 = R + 1 > 1$ and $p_2 = 1/S + 1 > 1$, we apply Hölder’s inequality and then consider (A1), (A2), (1.8) and (3.3) to deduce

$$\begin{aligned}
\int_0^1 |ag(z_t)| dx &= \int_{\Omega_1(t)} |ag(z_t)| dx + \int_{\Omega_2(t)} |ag(z_t)| dx \\
&\leq C_{10} \left(\int_{\Omega_1(t)} |ag(z_t)| |z_t|^{\frac{p_1-1}{R}} dx \right)^{\frac{1}{p_1}} + C_{10} \left(\int_{\Omega_2(t)} |ag(z_t)| |z_t|^{S(p_2-1)} dx \right)^{\frac{1}{p_2}} \\
&\leq C_{10} (-E')^{\frac{1}{R+1}} + C_{10} (-E')^{\frac{S}{S+1}}, \quad t \in [t_1, t_2].
\end{aligned} \tag{3.19}$$

Similarly as we just did for the obtention of (3.19), we can also deduce

$$\begin{aligned}
\int_0^1 |az_t|^2 dx &= \int_{\Omega_1(t)} |az_t|^2 dx + \int_{\Omega_2(t)} |az_t|^2 dx \\
&\leq C_{10} \int_{\Omega_1(t)} |ag(z_t)z_t|^{\frac{2}{R+1}} dx + C_{10} \int_{\Omega_2(t)} |ag(z_t)z_t| dx \\
&\leq C_{10} (-E')^{\frac{2}{R+1}} + C_{10} (-E'), \quad t \in [t_1, t_2].
\end{aligned} \tag{3.20}$$

Accordingly, combining (3.18)–(3.20) yield

$$\begin{aligned}
&\int_{t_1}^{t_2} E^{\alpha+1} dt + \int_{t_1}^{t_2} \int_0^1 E^\alpha \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt \leq C_{10} E(0)^\alpha E(t_1) \\
&+ C_{10} \int_{t_1}^{t_2} E^{\alpha+\frac{1}{2}} (-E')^{\frac{1}{R+1}} dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{S}{S+1}} dt + C_{10} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{1}{2}} dt.
\end{aligned} \tag{3.21}$$

Let us set $p_3 = 1/R + 1 > 1$ with $q_3 = R + 1 > 1$ and $p_4 = S + 1 > 1$ with $q_4 = 1/S + 1 > 1$, which satisfy that $1/p_3 + 1/q_3 = 1$ and $1/p_4 + 1/q_4 = 1$. Keeping in mind that (3.3) implies that $E(t_2) \leq E(t_1) \leq E(0)$ whenever $0 \leq t_1 < t_2 < \infty$, we apply Young's inequality to obtain

$$\begin{aligned}
\int_{t_1}^{t_2} E^{\alpha+\frac{1}{2}} (-E')^{\frac{1}{R+1}} dt &\leq \frac{1}{p_3 \varepsilon_1} \int_{t_1}^{t_2} E^{p_3(\alpha+\frac{1}{2})} dt + \frac{1}{q_3} \varepsilon_1^{\frac{q_3}{p_3}} \int_{t_1}^{t_2} (-E')^{\frac{q_3}{R+1}} dt \\
&\leq \frac{C_{10}}{\varepsilon_1} \int_{t_1}^{t_2} E^{(\frac{1}{R}+1)(\alpha+\frac{1}{2})} dt + C_{10} \varepsilon_1^R E(t_1), \quad \varepsilon_1 > 0,
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
\int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{S}{S+1}} dt &\leq \frac{1}{p_4 \varepsilon_2} \int_{t_1}^{t_2} E^{\alpha+1} E^{\frac{p_4}{2}-1} dt + \frac{1}{q_4} \varepsilon_2^{\frac{q_4}{p_4}} \int_{t_1}^{t_2} E^\alpha (-E')^{\frac{q_4 S}{S+1}} dt \\
&\leq \frac{C_{10}}{\varepsilon_2} E(0)^{\frac{S-1}{2}} \int_{t_1}^{t_2} E^{\alpha+1} dt + C_{10} \varepsilon_2^{\frac{1}{S}} E(0)^\alpha E(t_1), \quad \varepsilon_2 > 0,
\end{aligned} \tag{3.23}$$

$$\begin{aligned} \int_{t_1}^{t_2} E^\alpha E^{\frac{1}{2}} (-E')^{\frac{1}{2}} dt &\leq \frac{1}{2\varepsilon_1} \int_{t_1}^{t_2} E^{\alpha+1} dt + \frac{1}{2} \varepsilon_1 \int_{t_1}^{t_2} E^\alpha (-E') dt \\ &\leq \frac{1}{2\varepsilon_1} \int_{t_1}^{t_2} E^{\alpha+1} dt + \frac{1}{2} \varepsilon_1 E(0)^\alpha E(t_1), \quad \varepsilon_1 > 0. \end{aligned} \tag{3.24}$$

Taking into account the combination of (3.21) with (3.22)–(3.24), we infer from (3.22) that we need to ask $\alpha \geq 0$ so that $(1/R + 1)(\alpha + 1/2) = \alpha + 1$, which only happens when $\alpha = (R - 1)/2 \geq 0$. Let us choose $\varepsilon_1 = \varepsilon > 0$ and $\varepsilon_2 = \varepsilon E(0)^{\frac{S-1}{2}} > 0$. Then, from (3.21)–(3.24) it follows

$$\begin{aligned} &\int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + \int_{t_1}^{t_2} \int_0^1 E^{\frac{R-1}{2}} \left(\frac{1}{2} |z_{xx}|^2 + \frac{1}{2} |z_{xxx}|^2 \right) dx dt \\ &\leq \frac{C_{10}}{\varepsilon} \int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + C_{10} \left(E(0)^{\frac{R-1}{2}} + \varepsilon^R + \varepsilon^{\frac{1}{S}} E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + \varepsilon E(0)^{\frac{R-1}{2}} \right) E(t_1), \quad \varepsilon > 0. \end{aligned}$$

Here the choice of $\varepsilon = 2C_{10} > 0$ gives us a constant $C_{11} > 0$ such that

$$\int_{t_1}^{t_2} E^{\frac{R+1}{2}} dt + \int_{t_1}^{t_2} \int_0^1 E^{\frac{R-1}{2}} (|z_{xx}|^2 + |z_{xxx}|^2) dx dt \leq C_{11} \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right) E(t_1).$$

Finally, the previous inequality allows us to obtain (1.10) and (1.11) by letting $t_2 \rightarrow \infty$ and then applying (3.2) with

$$\alpha = \frac{R - 1}{2} \geq 0 \quad \text{and} \quad \beta = C_{11} \left(1 + E(0)^{\frac{S-1}{2S} + \frac{R-1}{2}} + E(0)^{\frac{R-1}{2}} \right) > 0.$$

The proof of Theorem 1.1 is complete. \square

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