Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 6485–6520

Journal of Differential Equations

www.elsevier.com/locate/jde

# On the cost of null controllability of a fourth-order parabolic equation

Nicolás Carreño, Patricio Guzmán *

*Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile*

Received 17 November 2015; revised 16 April 2016

Available online 6 September 2016

**Abstract**

In this paper, we study the cost of null controllability of a fourth-order parabolic equation. When the control time is large enough, we prove that this cost decreases exponentially to zero as the diffusion coefficient of the equation vanishes. When the control time is small, on the contrary, we prove that this cost increases exponentially to infinity.



*MSC:* 35B25; 35K35; 93B05; 93B07

*Keywords:* Fourth-order parabolic equation; Cost of null controllability; Boundary control; Uniform null controllability; Carleman estimates

## 1. Introduction

Let $T > 0$ and $L > 0$. Define the space $V := H^2 \cap H^1_0(0, L)$ and denote by $V^*$ its dual space. We identify $L^2(0, L)$ with itself to obtain $V \hookrightarrow L^2(0, L) \hookrightarrow V^*$, with each continuous injection being dense in the following.

* Corresponding author.
*E-mail addresses:* nicolas.carrenog@usm.cl (N. Carreño), patricio.guzman@alumnos.usm.cl (P. Guzmán).

http://dx.doi.org/10.1016/j.jde.2016.08.042

In this paper, we consider the fourth-order parabolic equation

$$\begin{cases} z_t + \varepsilon z_{xxxx} + M z_x = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0, & t \in (0,T), \\ z_{xx}(t,0) = u_2(t),\ z_{xx}(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ and $M \in \mathbb{R}$ are the diffusion and transport coefficients, respectively. The purpose of this paper is to study the cost of null controllability of equation (1.1). Being more precise, we are interested to know about its behavior with respect to the diffusion coefficient, and in particular, to know what happens as $\varepsilon \to 0^+$.

For a given space $Z$, the cost of null controllability of equation (1.1) is defined by

$$C_Z(\varepsilon) := \sup_{\substack{z_0 \in Z \\ z_0 \neq 0}} \min_{\substack{(u_1,u_2)\in L^2(0,T)^2 \\ z(T,\cdot)=0}} \frac{\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)}}{\|z_0\|^2_Z}. \tag{1.2}$$

Of course, this number for being well defined requires equation (1.1) to be well posed and null controllable, with initial conditions $z_0 \in Z$ and boundary controls $(u_1,u_2) \in L^2(0,T)^2$. Our first result addresses this problem and tells us that we can take $Z$ as $V^*$, or any subspace of it.

**Theorem 1.1.** *For every $z_0 \in V^*$, there exists $(u_1,u_2) \in L^2(0,T)^2$ such that the unique solution (defined by transposition) $z \in C([0,T];V^*)$ of equation (1.1) satisfies $z(T,\cdot) = 0$ in $V^*$. Moreover, there exists $C > 0$, independent of $(T,L,\varepsilon) \in (0,+\infty)^3$ and $M \in \mathbb{R}$, such that*

$$\begin{aligned} &\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} \\ &\quad \leq \frac{C(L^3+L^7)}{T\varepsilon^2 \min\{L^4, L^2\pi^2, \pi^4\}} \exp\left\{ C\left( \frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}} + \frac{L|M|^{1/3}}{\varepsilon^{1/3}} + \frac{TL^2|M|^2}{\varepsilon} \right) \right\} \|z_0\|^2_{V^*}. \end{aligned} \tag{1.3}$$

The proof of this result relies on the controllability–observability duality (see [7, Theorem 2.44], [21, Remark 13.1] or [25, Theorem 11.2.1] for instance), and hence, we just focus on the obtention of an observability inequality for the adjoint equation associated to equation (1.1), which is

$$\begin{cases} -q_t + \varepsilon q_{xxxx} - M q_x = 0, & (t,x) \in (0,T) \times (0,L), \\ q(t,0) = 0,\ q(t,L) = 0, & t \in (0,T), \\ q_{xx}(t,0) = 0,\ q_{xx}(t,L) = 0, & t \in (0,T), \\ q(T,x) = q_T(x), & x \in (0,L). \end{cases} \tag{1.4}$$

The main tool to obtain such an observability inequality is a Carleman estimate for adjoint equation (1.4). In the context of fourth-order parabolic equations, these estimates have been used for studying problems of internal control [6,14–16,26], boundary control [2,5] and stability of inverse problems [1,15,20]. However, none of these estimates is suitable for us because of the boundary conditions of adjoint equation (1.4). Hence, we derive a new Carleman estimate (see Proposition 3.1), which is optimal with respect to the existent ones in the sense that we use the optimal weight functions (see Remark 3.3).

Another consequence of Theorem 1.1 is an upper bound for $C_{V^*}(\varepsilon)$, namely

$$C_{V^*}(\varepsilon) \leq \frac{C(L^3+L^7)}{T\varepsilon^2 \min\{L^4, L^2\pi^2, \pi^4\}} \exp\left\{C\left(\frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}} + \frac{L|M|^{1/3}}{\varepsilon^{1/3}} + \frac{TL^2|M|^2}{\varepsilon}\right)\right\}. \tag{1.5}$$

Nevertheless, this information does not allow us to say anything about the behavior of $C_{V^*}(\varepsilon)$ as $\varepsilon \to 0^+$. We succeeded in saying something for $C_{L^2(0,L)}(\varepsilon)$, but not from (1.5). In order to understand the behavior we expect to find for $C_{L^2(0,L)}(\varepsilon)$ as $\varepsilon \to 0^+$, let us consider the control properties of the transport equation

$$\begin{cases} z_t + Mz_x = 0, & (t,x) \in (0,T) \times (0,L), \\ z(0,x) = z_0(x), & x \in (0,L), \end{cases}$$

with the boundary conditions

$$\begin{cases} z(t,0) = u(t), & t \in (0,T), \quad \text{if } M > 0, \\ z(t,L) = u(t), & t \in (0,T), \quad \text{if } M < 0. \end{cases}$$

It is known (see [7, Theorem 2.6] for instance) that this equation is null controllable if and only if $T \geq L/|M|$. Indeed, for any $z_0 \in L^2(0,L)$, it suffices to take the boundary control $u(\cdot) = 0$ to obtain $z(T,\cdot) = 0$ in $L^2(0,L)$. In virtue of this fact and Theorem 1.1, we expect that $C_{L^2(0,L)}(\varepsilon)$ would decrease to zero as $\varepsilon \to 0^+$ when $T \geq L/|M|$. Furthermore, when $T < L/|M|$, on the contrary, we expect that $C_{L^2(0,L)}(\varepsilon)$ would increase to infinity. This kind of problem was initially considered in [8] for the case of the heat equation with vanishing viscosity coefficient. Later, improvements have been done in [17,22,23]. For similar results concerning the linear Korteweg–de Vries equation with vanishing dispersion coefficient, we refer to [3,18,19]. Up to our knowledge, this is the first work addressing this kind of problem for a fourth-order parabolic equation.

In the following result, we establish the uniform null controllability, with respect to the diffusion coefficient, of equation (1.1) when the control time is large enough and the initial condition is in $L^2(0,L)$.

**Theorem 1.2.** *Let $T \geq 40L/|M|$ with $M \in \mathbb{R} - \{0\}$. For every $z_0 \in L^2(0,L)$, there exists $(u_1,u_2) \in L^2(0,T)^2$ such that the unique solution (defined by transposition) $z \in C([0,T];V^*)$ of equation (1.1) satisfies $z(T,\cdot) = 0$ in $V^*$. Moreover, there exist $C_1 > 0$ and $C_2 > 0$, both independent of $(T,L,\varepsilon) \in (0,+\infty)^3$ and $M \in \mathbb{R} - \{0\}$, such that*

$$\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} \leq \frac{C_1}{\varepsilon^2}\left(\frac{L^3+L^7}{T}\right) \exp\left\{-C_2 \frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\} \|z_0\|^2_{L^2(0,L)}. \tag{1.6}$$

From (1.2) and (1.6) we deduce that $C_{L^2(0,L)}(\varepsilon)$ decreases exponentially to zero as the diffusion coefficient vanishes.

**Corollary 1.3.** *Let $T \geq 40L/|M|$ with $M \in \mathbb{R} - \{0\}$. Then, $\lim_{\varepsilon \to 0^+} C_{L^2(0,L)}(\varepsilon) = 0$.*

The proof of Theorem 1.2 follows the lines of [19], that is, we prove an observability inequality for adjoint equation (1.4) via the combination of an exponential dissipation result (see

Proposition 4.1) and a suitable Carleman estimate, which actually is the one used to prove Theorem 1.1. Here, the optimality of that estimate is essential, not as in the proof of Theorem 1.1.

In the following result, we give a lower bound for the norms of the controls when the control time is small and the initial condition is in $L^2(0, L)$.

**Theorem 1.4.** *Let $T < L/|M|$ with $M \in \mathbb{R} - \{0\}$. There exist $C_1 > 0$, $C_2 > 0$ and $\varepsilon_0 > 0$ (independent of $\varepsilon > 0$) and $z_0 \in V$ such that for any $(u_1, u_2) \in L^2(0, T)^2$ driving the unique solution (defined by transposition) $z \in C([0, T]; V^*)$ of equation (1.1) to the null state satisfies*

$$\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} \geq C_1 \frac{\varepsilon^3}{\varepsilon^5 + 1} \exp\left\{\frac{C_2}{T^{1/3}\varepsilon^{1/3}}\right\} \|z_0\|^2_{L^2(0,L)}, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{1.7}$$

From (1.2) and (1.7) we deduce that $C_{L^2(0,L)}(\varepsilon)$, and hence $C_{V^*}(\varepsilon)$, increases exponentially to infinity as the diffusion coefficient vanishes.

**Corollary 1.5.** *Let $T < L/|M|$ with $M \in \mathbb{R} - \{0\}$. Then, $\lim_{\varepsilon \to 0^+} C_{L^2(0,L)}(\varepsilon) = \lim_{\varepsilon \to 0^+} C_{V^*}(\varepsilon) = +\infty$.*

The result given by Theorem 1.4 is similar to the one proved in [8, Theorem 2] for the case of the heat equation with vanishing viscosity coefficient. There, it is deduced that their cost of null controllability explodes as $\varepsilon \to 0^+$ when $T < L/M$ if $M > 0$ and $T < 2L/|M|$ if $M < 0$. The latter has been improved to $T < 2\sqrt{2}L/|M|$ in [23, Theorem 1.3]. In view of this result, it is natural to expect in our case a similar asymmetry with respect to the sign of $M \in \mathbb{R} - \{0\}$. Since the same analysis cannot be applied to our case due to the structure of adjoint equation (1.4), we adapt the arguments introduced in [19], which later were applied in [3].

**Remark 1.6.** It would be interesting to study the cost of null controllability of equation (1.1) when only one boundary control is considered. Nevertheless, the question of null controllability for that equation is an open problem. For instance, we could try to address that problem as in [4,6] by using the moment theory developed by Fattorini and Russell in [11]. However, we were not able to diagonalize the underlying spatial operator in equation (1.1) (see [17, Section 2.2]). Furthermore, this is the reason why the arguments employed in [8,17,23] are not suitable for us to prove Theorem 1.4.

This paper is organized as follows. In Section 2, we prove the well-posedness results needed throughout this paper. In Section 3, we prove Theorem 1.1, which states the null controllability of equation (1.1). In Sections 4.1 and 4.2, we prove Theorems 1.2 and 1.4, respectively, which are the results related to the cost of null controllability of equation (1.1). In Section 5, we comment about the extension of our results to other boundary conditions in equation (1.1). Finally, in Appendix A, we prove the new Carleman estimate.

## 2. Well-posedness

In this section, we present the well-posedness results needed for the study of equation (1.1). To this end, let us consider the equation

$$\begin{cases} z_t + \varepsilon z_{xxxx} + M z_x = f, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0, & t \in (0,T), \\ z_{xx}(t,0) = u_2(t),\ z_{xx}(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{2.1}$$

### 2.1. Finite energy solutions

In this section, we derive some well-posedness results for equation (2.1) with regular enough data. We begin by applying the Faedo–Galerkin method to treat the case of homogeneous boundary conditions. To this end, let us define $A : D(A) \subset L^2(0,L) \to L^2(0,L)$ by $Aw := w''''$ with

$$D(A) := \left\{ w \in H^4(0,L) \ / \ w(0) = w''(0) = 0,\ w(L) = w''(L) = 0 \right\}.$$

This self-adjoint operator has a compact resolvent due to Rellich's Theorem, and hence, it has a discrete spectrum consisting only in real eigenvalues, $\{\lambda_n\}_{n\in\mathbb{N}}$, with its corresponding eigenfunctions, $\{w_n\}_{n\in\mathbb{N}}$, forming an orthonormal basis of $L^2(0,L)$. In this case, it is known that the eigenvalues and eigenfunctions are respectively given by

$$\lambda_n = \frac{\pi^4 n^4}{L^4}, \quad w_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n}{L} x\right), \quad n \in \mathbb{N}.$$

Since the continuous injection $V \hookrightarrow L^2(0,L)$ is also compact according to Rellich's Theorem, we have that $\{w_n\}_{n\in\mathbb{N}}$ form an orthogonal basis of $V$ (see [9, Theorem 7, p. 39] for instance).

**Proposition 2.1.** *Let $f \in L^2(0,T;L^2(0,L))$ and $u_1 = u_2 = 0$.*

*(a) If $z_0 \in L^2(0,L)$, then equation (2.1) has a unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;V)$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;V)} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{L^2(0,L)}\right). \tag{2.2}$$

*(b) If $z_0 \in V$, then equation (2.1) has a unique solution $z \in C([0,T];V) \cap L^2(0,T;D(A))$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;V)\cap L^2(0,T;D(A))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_V\right). \tag{2.3}$$

**Proof.** The proof consists in applying the Faedo–Galerkin method. We just derive (2.2) and (2.3) and leave the details, that may be found in [10, Chapter 7] or [24, Chapter 2, Section 3] for instance, to the reader. In what follows, we are going to use that $\|w^{(n)}\|_{L^2(0,L)} + \|w\|_{L^2(0,L)}$, with $n \in \mathbb{N}$, is a norm equivalent to the norm $\|w\|_{H^n(0,L)}$.

Let $t \in [0,T]$. On the one hand, multiplying equation (2.1) by $z = z(t,x)$, then performing some integrations by parts on $(0,L)$ and finally applying the Cauchy–Schwarz inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |z|^2 dx\right) + \varepsilon \int_0^L |z_{xx}|^2 dx \le \|f(t,\cdot)\|_{L^2(0,L)} \|z(t,\cdot)\|_{L^2(0,L)}.$$

Here we integrate on $(0,t)$ and then employ the Cauchy inequality to conclude

$$\|z(t,\cdot)\|^2_{L^2(0,L)} + 2\varepsilon\int_0^t \|z_{xx}(\tau,\cdot)\|^2_{L^2(0,L)}d\tau \leq 2\|f\|^2_{L^1(0,T;L^2(0,L))} + \frac{1}{2}\|z\|^2_{L^\infty(0,T;L^2(0,L))} + \|z_0\|^2_{L^2(0,L)}.$$

The previous inequality allows us to obtain (2.2), but more precisely

$$\|z\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;V)} \leq C\left(\|f\|_{L^1(0,T;L^2(0,L))} + \|z_0\|_{L^2(0,L)}\right). \tag{2.4}$$

On the other hand, multiplying equation (2.1) by $z_{xxxx} = z_{xxxx}(t,x)$, then performing some integrations by parts on $(0,L)$ and finally applying the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |z_{xx}|^2dx\right) + \frac{\varepsilon}{2}\int_0^L |z_{xxxx}|^2dx \leq \frac{1}{\varepsilon}\int_0^L |f|^2dx + \frac{|M|^2}{\varepsilon}\int_0^L |z_x|^2dx.$$

Here we plug (2.2) to deduce (2.3). The proof of Proposition 2.1 is complete. □

The next result is inspired by [12, Remark 1]. It will be needed for the new Carleman estimate (see Remark 3.2).

**Corollary 2.2.** *Let $f \in L^2(0,T;L^2(0,L))$, $u_1 = u_2 = 0$ and $z_0 \in L^2(0,L)$. Consider $\gamma \in H^1(0,T)$ such that $\gamma(0) = 0$. Then, the unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;V)$ of equation (2.1) satisfies $\gamma z \in C([0,T];V) \cap L^2(0,T;D(A))$.*

**Proof.** Let $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;V)$ be the unique solution of equation (2.1) given by Proposition 2.1 (a). Then, thanks to equation (2.1), we have that $p(t,x) := \gamma(t)z(t,x)$ satisfies the equation

$$\begin{cases} p_t + \varepsilon p_{xxxx} + Mp_x = f + \gamma' z, & (t,x) \in (0,T)\times(0,L), \\ p(t,0) = 0,\ p(t,L) = 0, & t \in (0,T), \\ p_{xx}(t,0) = 0,\ p_{xx}(t,L) = 0, & t \in (0,T), \\ p(0,x) = 0, & x \in (0,L). \end{cases} \tag{2.5}$$

Therefore, taking into account that $f + \gamma' z \in L^2(0,T;L^2(0,L))$, the result follows by applying Proposition 2.1 (b) to equation (2.5). The proof of Corollary 2.2 is complete. □

The proof of the next result also consists in applying the Faedo–Galerkin method.

**Proposition 2.3.** *Let $f \in L^1(0,T;V)$, $u_1 = u_2 = 0$ and $z_0 \in V$. Then, equation (2.1) has a unique solution $z \in C([0,T];V) \cap L^2(0,T;D(A))$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;V)\cap L^2(0,T;D(A))} \leq C\left(\|f\|_{L^1(0,T;V)} + \|z_0\|_V\right). \tag{2.6}$$

**Proof.** As in the proof of Proposition 2.1, we just derive (2.6). Let $t \in [0,T]$. Multiplying equation (2.1) by $z_{xxxx} = z_{xxxx}(t,x)$, then performing some integrations by parts on $(0,L)$ and finally applying the Cauchy inequality, we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |z_{xx}|^2 dx\right) + \frac{\varepsilon}{2}\int_0^L |z_{xxxx}|^2 dx \le \int_0^L f_{xx} z_{xx}\, dx + \frac{|M|^2}{2\varepsilon}\int_0^L |z_x|^2 dx.$$

Here we integrate on $(0,t)$ and then employ the Cauchy–Schwarz and Cauchy inequalities to conclude

$$\|z_{xx}(t,\cdot)\|^2_{L^2(0,L)} + \varepsilon\int_0^t \|z_{xxxx}(\tau,\cdot)\|^2_{L^2(0,L)} d\tau \le 2\|f\|^2_{L^1(0,T;V)} + \frac{1}{2}\|z_{xx}\|^2_{L^\infty(0,T;L^2(0,L))} + \|z_0\|^2_V + \frac{|M|^2}{\varepsilon}\|z\|^2_{L^2(0,T;V)}.$$

Therefore, we deduce (2.6) by plugging (2.4) into the previous inequality. The proof of Proposition 2.3 is complete. □

In order to treat the case of non-homogeneous boundary conditions, we employ a suitable lifting function together with Proposition 2.1 (a). To this end, let us introduce the space

$$H^1_r(0,L) := \left\{ w \in H^1(0,L) \ / \ w(L) = 0 \right\}.$$

**Proposition 2.4.** *Let $f \in L^2(0,T;L^2(0,L))$, $(u_1,u_2) \in H^1(0,T)^2$ and $z_0 \in L^2(0,L)$. Then, equation (2.1) has a unique solution $z \in C([0,T];L^2(0,L)) \cap L^2(0,T;H^2 \cap H^1_r(0,L))$. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;H^2\cap H^1_r(0,L))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{H^1(0,T)^2} + \|z_0\|_{L^2(0,L)}\right). \quad (2.7)$$

**Proof.** Consider the lifting function

$$\psi(t,x) := u_1(t)\left(-\frac{1}{L}x + 1\right) + u_2(t)\left(-\frac{1}{6L}x^3 + \frac{1}{2}x^2 - \frac{L}{3}x\right).$$

Taking into account that $g := f - \psi_t - \varepsilon\psi_{xxxx} - M\psi_x$ and $y_0(x) := z_0(x) - \psi(0,x)$ are elements of $L^2(0,T;L^2(0,L))$ and $L^2(0,L)$, respectively, it follows that the equation

$$\begin{cases} y_t + \varepsilon y_{xxxx} + M y_x = g, & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = 0, \ \ y(t,L) = 0, & t \in (0,T), \\ y_{xx}(t,0) = 0, \ \ y_{xx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L), \end{cases}$$

has a unique solution $y \in C([0,T]; L^2(0,L)) \cap L^2(0,T;V)$ thanks to Proposition 2.1 (a). Furthermore, in view of (2.2), that solution satisfies

$$\|y\|_{L^\infty(0,T;L^2(0,L))\cap L^2(0,T;V)} \le C\left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)}\right). \tag{2.8}$$

From $\psi(t,0) = u_1(t)$ and $\psi_{xx}(t,0) = u_2(t)$, we conclude that $z := y + \psi$, which belongs to $C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2 \cap H^1_r(0,L))$, is a solution of equation (2.1). Because of the continuous injection $H^1(0,T) \hookrightarrow L^\infty(0,T)$, we can get the inequalities

$$\|g\|_{L^2(0,T;L^2(0,L))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(u_1,u_2)\|_{H^1(0,T)^2}\right),$$
$$\|y_0\|_{L^2(0,L)} \le C\left(\|z_0\|_{L^2(0,L)} + \|(u_1,u_2)\|_{H^1(0,T)^2}\right),$$

which combined with $\|z\| - \|\psi\| \le \|y\|$ (valid for any norm) and (2.8) lead us to (2.7). That inequality and the linearity of the equation yield the uniqueness of solutions. The proof of Proposition 2.4 is complete. □

*2.2. Solutions defined by transposition*

With the aim to motivate a definition of solution for equation (1.1), given the data $(u_1,u_2) \in L^2(0,T)^2$ and $z_0 \in V^*$, let us consider the next formal computations. After the change of variable $t \to T - t$, Proposition 2.3 tells us that the equation

$$\begin{cases} -q_t + \varepsilon q_{xxxx} - Mq_x = g, & (t,x) \in (0,T)\times(0,L), \\ q(t,0) = 0,\ q(t,L) = 0, & t \in (0,T), \\ q_{xx}(t,0) = 0,\ q_{xx}(t,L) = 0, & t \in (0,T), \\ q(T,x) = 0, & x \in (0,L), \end{cases} \tag{2.9}$$

has a unique solution $q \in C([0,T];V) \cap L^2(0,T;D(A))$ for any $g \in L^1(0,T;V)$. Therefore, multiplying equation (1.1) by $q = q(t,x)$ and then performing some integrations by parts, we get

$$\int_0^T\int_0^L z(t,x)\left(-q_t(t,x) + \varepsilon q_{xxxx}(t,x) - Mq_x(t,x)\right)dx\,dt + \int_0^L z(t,x)q(t,x)\Big|_{t=0}^{t=T} dx$$
$$- \varepsilon\int_0^T z(t,x)q_{xxx}(t,x)\Big|_{x=0}^{x=L} dt - \varepsilon\int_0^T z_{xx}(t,x)q_x(t,x)\Big|_{x=0}^{x=L} dt = 0. \tag{2.10}$$

Note that all the terms related to $q = q(t,x)$ are well defined due to its regularity and the continuous injection $H^4(0,L) \hookrightarrow C^3([0,L])$. In order to give a sense to the previous formal computations, and keeping in mind the regularity of $q = q(t,x)$, we present the following definition.

**Definition 2.1.** Let $(u_1,u_2) \in L^2(0,L)^2$ and $z_0 \in V^*$. We say that $z = z(t,x)$ is a solution defined by transposition of equation (1.1) if $z \in C([0,T];V^*)$ is such that for every $g \in L^1(0,T;V)$ it satisfies

$$\langle z, g\rangle_{L^\infty(0,T;V^*)\times L^1(0,T;V)} = \langle z_0, q(0,\cdot)\rangle_{V^*\times V} - \varepsilon\int_0^T u_1(t)q_{xxx}(t,0)\,dt - \varepsilon\int_0^T u_2(t)q_x(t,0)\,dt, \quad (2.11)$$

with $q = q(t,x)$ being the unique solution of equation (2.9) given by Proposition 2.3.

The results developed in this section allow us to justify the existence and uniqueness of solutions defined by transposition for equation (1.1).

**Proposition 2.5.** *Let $(u_1, u_2) \in L^2(0,L)^2$ and $z_0 \in V^*$. Then, equation (1.1) has a unique solution $z \in C([0,T]; V^*)$ defined by transposition. Moreover, there exists $C > 0$ such that*

$$\|z\|_{L^\infty(0,T;V^*)} \le C\left(\|(u_1,u_2)\|_{L^2(0,T)^2} + \|z_0\|_{V^*}\right). \quad (2.12)$$

**Proof.** For the moment, let us assume that $(u_1, u_2) \in H^1(0,L)^2$ and $z_0 \in L^2(0,L)$, so that equation (1.1) would have a unique solution $z \in C([0,T]; L^2(0,L)) \cap L^2(0,T; H^2 \cap H_r^1(0,L))$ due to Proposition 2.4. Note that in particular we have $z \in C([0,T]; V^*)$.

Let $g \in L^1(0,T;V)$ and take $q \in C([0,T];V) \cap L^2(0,T;D(A))$, the corresponding unique solution of equation (2.9) given by Proposition 2.3. Multiplying equation (1.1) by $q = q(t,x)$ and then performing some integrations by parts, we see that $z \in C([0,T]; V^*)$ satisfies (2.11).

In (2.11) we employ the Cauchy–Schwarz inequality to obtain

$$\left|\langle z, g\rangle_{L^\infty(0,T;V^*)\times L^1(0,T;V)}\right| \le C\left(\|z_0\|_{V^*}\|q(0,\cdot)\|_V + \|u_1\|_{L^2(0,T)}\|q_{xxx}(\cdot,0)\|_{L^2(0,T)} + \|u_2\|_{L^2(0,T)}\|q_x(\cdot,0)\|_{L^2(0,T)}\right).$$

Hence, the continuous injection $H^1(0,L) \hookrightarrow L^\infty(0,L)$ together with (2.6) leads us to

$$\left|\langle z, g\rangle_{L^\infty(0,T;V^*)\times L^1(0,T;V)}\right| \le C\left(\|(u_1,u_2)\|_{L^2(0,T)^2} + \|z_0\|_{V^*}\right)\|g\|_{L^1(0,T;V)},$$

from which we deduce (2.12). Since the continuous injection $L^2(0,L) \hookrightarrow V^*$ is dense, we can use (2.11), (2.12) and a density argument to conclude that equation (1.1) has a unique solution $z \in C([0,T]; V^*)$ defined by transposition provided that $(u_1,u_2) \in L^2(0,T)^2$ and $z_0 \in V^*$. The proof of Proposition 2.5 is complete. □

## 3. Null controllability

This section is devoted to the proof of Theorem 1.1. The main tool of the proof is a Carleman estimate for adjoint equation (1.4). Let us introduce the weight functions

$$\alpha(t,x) := \beta(t)\left(-\frac{1}{2}x^2 + 8Lx + \frac{L^2}{2}\right),$$

$$\beta(t) = \frac{1}{t^{1/3}(T-t)^{1/3}}, \quad (t,x) \in [0,T]\times[0,L]. \quad (3.1)$$

Following a procedure described in [13, Chapter I] due to Fursikov and Imanuvilov, we can obtain the next Carleman estimate, whose long and technical proof is given in Appendix A.

**Proposition 3.1.** *There exists $C > 0$, independent of $(T, L, \varepsilon) \in (0, +\infty)^3$ and $M \in \mathbb{R}$, such that for any*

$$s \geq \frac{9}{40} T^{1/3} L^{-2/3} \varepsilon^{-1/3} + \frac{1}{8} T^{2/3} L^{-1} \varepsilon^{-1/3} |M|^{1/3} + C T^{2/3} L^{-2}, \tag{3.2}$$

*we have that the unique solution $q = q(t, x)$ of adjoint equation (1.4) with $q_T \in L^2(0, L)$ satisfies*

$$\int_0^T \int_0^L e^{-2s\alpha} \left( s^7 L^6 \beta^7 |q|^2 + s^5 L^4 \beta^5 |q_x|^2 + s^3 L^2 \beta^3 |q_{xx}|^2 + s\beta |q_{xxx}|^2 \right) dx dt$$
$$\leq C \int_0^T e^{-2s\alpha(t,0)} \left( s^5 L^5 \beta(t)^5 |q_x(t, 0)|^2 + s L \beta(t) |q_{xxx}(t, 0)|^2 \right) dt. \tag{3.3}$$

**Remark 3.2.** Every term in (3.3) is well defined in virtue of Corollary 2.2. Indeed, it suffices to take $\gamma(t) = e^{-s\alpha(t,0)} \beta(t)^{1/2}$. Then, since $\alpha(t, x) \geq \alpha(t, 0)$ for every $(t, x) \in [0, T] \times [0, L]$, we conclude that $e^{-s\alpha} \beta^{1/2} q_{xxx} \in L^2(0, T; L^2(0, L))$. Finally, $e^{-s\alpha(\cdot,0)} \beta^{1/2} q_{xxx}(\cdot, 0) \in L^2(0, T)$ is a consequence of the continuous injection $H^4(0, L) \hookrightarrow C^3([0, L])$.

**Remark 3.3.** From the proof of Proposition 3.1 in Appendix A, we can directly check that (3.3) remains valid, up to different constants in (3.2), if we take $\beta(t) = t^{-m}(T - t)^{-m}$ with $m \geq 1/3$. For this reason we claim that our estimate is the optimal one. Most of the existent Carleman estimates for fourth-order parabolic equations use $m = 1$ [1,5,6,14–16,20,26]. However, in [2] it is used $m \geq 2/5$.

**Remark 3.4.** From the proof of Proposition 3.1 in Appendix A, we can directly check that (3.3) remains valid if we change the boundary conditions of adjoint equation (1.4) to $q(t, 0) = q(t, L) = 0$ and $q_x(t, 0) = q_x(t, L) = 0$ for $t \in (0, T)$. Nevertheless, the constant $C > 0$ in (3.2) might differ.

The next result is a consequence of this Carleman estimate and it will be useful to obtain the required observability inequality to prove Theorem 1.1. Furthermore, in Section 4, it will be useful for the study of the cost of null controllability of equation (1.1).

**Lemma 3.5.** *Let $n \in \{0, 1, 2, 3\}$. There exists $C > 0$, independent of $(T, L, \varepsilon) \in (0, +\infty)^3$ and $M \in \mathbb{R}$, such that for any*

$$s \geq \frac{9}{40} T^{1/3} L^{-2/3} \varepsilon^{-1/3} + \frac{1}{8} T^{2/3} L^{-1} \varepsilon^{-1/3} |M|^{1/3} + C T^{2/3} L^{-2},$$

*we have that the unique solution $q = q(t, x)$ of adjoint equation (1.4) with $q_T \in V$ satisfies*

$$\int_{2T/3}^{3T/4}\int_0^L \left|\frac{\partial^n q}{\partial x^n}\right|^2 dxdt$$

$$\leq C\left(L^{3-2n}+L^{7-2n}\right)\exp\left\{27sT^{-2/3}L^2\right\}\int_0^T \left(|q_x(t,0)|^2+|q_{xxx}(t,0)|^2\right)dt. \quad (3.4)$$

**Proof.** In Proposition 3.1 we obtained

$$\int_{2T/3}^{3T/4}\int_0^L e^{-2s\alpha}s^{7-2n}L^{6-2n}\beta^{7-2n}\left|\frac{\partial^n q}{\partial x^n}\right|^2 dxdt$$

$$\leq C\int_0^T e^{-2s\alpha(t,0)}\left(s^5L^5\beta(t)^5|q_x(t,0)|^2+sL\beta(t)|q_{xxx}(t,0)|^2\right)dt. \quad (3.5)$$

To obtain the desired result, we are going to bound from below and above the quantities in (3.5). On the one hand, for $(t,x)\in[2T/3,3T/4]\times[0,L]$ we have the inequalities $(9/2)^{1/3}T^{-2/3}\leq\beta(t)\leq(16/3)^{1/3}T^{-2/3}$ and $L^2/2\leq(-1/2)x^2+8Lx+(L^2/2)\leq 8L^2$, which imply

$$\exp\left\{-16\left(\frac{16}{3}\right)^{1/3}sT^{-2/3}L^2\right\}s^{7-2n}L^{6-2n}\left(\frac{9}{2}\right)^{\frac{7-2n}{3}}T^{\frac{-14+4n}{3}}\int_{2T/3}^{3T/4}\int_0^L\left|\frac{\partial^n q}{\partial x^n}\right|^2 dxdt$$

$$\leq\int_{2T/3}^{3T/4}\int_0^L e^{-2s\alpha}s^{7-2n}L^{6-2n}\beta^{7-2n}\left|\frac{\partial^n q}{\partial x^n}\right|^2 dxdt.$$

Hence, considering this inequality in (3.5) yields

$$\exp\left\{-16\left(\frac{16}{3}\right)^{1/3}sT^{-2/3}L^2\right\}s^{7-2n}L^{6-2n}\left(\frac{9}{2}\right)^{\frac{7-2n}{3}}T^{\frac{-14+4n}{3}}\int_{2T/3}^{3T/4}\int_0^L\left|\frac{\partial^n q}{\partial x^n}\right|^2 dxdt$$

$$\leq C\int_0^T e^{-2s\alpha(t,0)}\left(s^5L^5\beta(t)^5|q_x(t,0)|^2+sL\beta(t)|q_{xxx}(t,0)|^2\right)dt. \quad (3.6)$$

On the other hand, let $t\in[0,T]$. The inequality $e^x\geq x^m/m!$, which holds true for $x\geq 0$ and $m\in\mathbb{N}$, allows us to deduce that $e^{(3/2)s\alpha(t,0)}\left(sL^2\beta(t)/4\right)^m/m!\leq e^{2s\alpha(t,0)}$, which combined with $4^{1/3}T^{-2/3}\leq\beta(t)$ gives us

$$e^{-2s\alpha(t,0)}s^m L^m \beta(t)^m \leq \frac{4^m m!}{L^m}\exp\left\{-\frac{4^{1/3}3}{4}sT^{-2/3}L^2\right\}.$$

Hence, the application of this inequality with $m=1$ and $m=5$ leads us to

$$\int_0^T e^{-2s\alpha(t,0)}\left(s^5L^5\beta(t)^5|q_x(t,0)|^2 + sL\beta(t)|q_{xxx}(t,0)|^2\right)dt \leq$$
$$\left(\frac{4^5 120}{L^5}+\frac{4}{L}\right)\exp\left\{-\frac{4^{1/3}3}{4}sT^{-2/3}L^2\right\}\int_0^T\left(|q_x(t,0)|^2+|q_{xxx}(t,0)|^2\right)dt. \quad (3.7)$$

Finally, by plugging (3.7) into (3.6) and then using that $s^{7-2n}L^{6-2n}T^{\frac{-14+4n}{3}} \geq CL^{-8+2n}$ is a consequence of $s \geq CT^{2/3}L^{-2}$, we arrive at (3.4). From the regularity of $q=q(t,x)$ given by Proposition 2.1 (b), we conclude that both $q_x(t,0)$ and $q_{xxx}(t,0)$ are well defined as elements of $L^2(0,T)$ in virtue of the continuous injection $H^4(0,L) \hookrightarrow C^3([0,L])$. The proof of Lemma 3.5 is complete. □

We proceed to prove the null controllability of equation (1.1).

**Proof of Theorem 1.1.** In virtue of the controllability–observability duality, we need to prove that the unique solution $q=q(t,x)$ of adjoint equation (1.4) with $q_T \in V$ satisfies

$$\|q(0,\cdot)\|_V^2 \leq K(T,L,\varepsilon,M)\int_0^T\left(|q_x(t,0)|^2+|q_{xxx}(t,0)|^2\right)dt,$$

where

$$K(T,L,\varepsilon,M) := \frac{C(L^3+L^7)}{T\min\{L^4,L^2\pi^2,\pi^4\}}\exp\left\{C\left(\frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}}+\frac{L|M|^{1/3}}{\varepsilon^{1/3}}+\frac{TL^2|M|^2}{\varepsilon}\right)\right\}.$$

Note that the controllability–observability duality also implies an upper bound for the norm of the controls, namely

$$\|u_1\|_{L^2(0,T)}^2 + \|u_2\|_{L^2(0,T)}^2 \leq \frac{K(T,L,\varepsilon,M)}{\varepsilon^2}\|z_0\|_{V^*}^2.$$

We have that the unique solution $q \in C([0,T];V)\cap L^2(0,T;D(A))$ of adjoint equation (1.4), given by Proposition 2.1 (b), satisfies

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^L |q_{xx}|^2dx\right)+\frac{\varepsilon}{2}\int_0^L |q_{xxxx}|^2dx \leq \frac{|M|^2}{2\varepsilon}\int_0^L |q_x|^2dx. \quad (3.8)$$

In order to handle the right-hand side of (3.8), we are going to use the Poincaré inequality. Since for every $t \in [0, T]$ we have that $q(t, \cdot) \in V$, it holds

$$\int_0^L |q(t,x)|^2 dx \le \frac{L^2}{\pi^2} \int_0^L |q_x(t,x)|^2 dx. \tag{3.9}$$

Furthermore, after one integration by parts and the use of the Cauchy inequality, we see that for any $\delta > 0$ it is valid

$$\int_0^L |q_x(t,x)|^2 dx = -\int_0^L q(t,x) q_{xx}(t,x)\, dx \le \frac{1}{2\delta} \int_0^L |q(t,x)|^2 dx + \frac{\delta}{2} \int_0^L |q_{xx}(t,x)|^2 dx.$$

In this inequality we plug (3.9) and then choose $\delta = L^2/\pi^2$ to obtain

$$\int_0^L |q_x(t,x)|^2 dx \le \frac{L^2}{\pi^2} \int_0^L |q_{xx}(t,x)|^2 dx. \tag{3.10}$$

Therefore, (3.8) together with (3.10) allows us to get

$$\frac{d}{dt}\left( \exp\left\{ \frac{|M|^2 L^2}{\pi^2 \varepsilon} t \right\} \int_0^L |q_{xx}|^2 dx \right) \ge 0,$$

which implies

$$\|q_{xx}(0,\cdot)\|^2_{L^2(0,L)} \le \frac{12}{T} \exp\left\{ \frac{|M|^2 L^2}{\pi^2 \varepsilon} \frac{3T}{4} \right\} \int_{2T/3}^{3T/4} \int_0^L |q_{xx}|^2 dx dt.$$

Note that in virtue of (3.9) and (3.10), we actually have

$$\min\left\{ \frac{1}{3}, \frac{\pi^2}{3L^2}, \frac{\pi^4}{3L^4} \right\} \|q(0,\cdot)\|^2_V \le \frac{12}{T} \exp\left\{ \frac{|M|^2 L^2}{\pi^2 \varepsilon} \frac{3T}{4} \right\} \int_{2T/3}^{3T/4} \int_0^L |q_{xx}|^2 dx dt.$$

Finally, (1.3) is a consequence of the previous inequality, Lemma 3.5 applied with $n = 2$ and the choice of

$$s = \frac{9}{40} T^{1/3} L^{-2/3} \varepsilon^{-1/3} + \frac{1}{8} T^{2/3} L^{-1} \varepsilon^{-1/3} |M|^{1/3} + C T^{2/3} L^{-2}.$$

The proof of Theorem 1.1 is complete. □

## 4. Cost of null controllability

This section is devoted to the proof of Theorems 1.2 and 1.4. In order to prove these theorems, we need to deduce some properties for the solutions of adjoint equation (1.4).

### 4.1. Uniform cost

The purpose of this section is to prove Theorem 1.2. We begin by obtaining an exponential dissipation result for adjoint equation (1.4).

**Proposition 4.1.** *Let $M \in \mathbb{R} - \{0\}$. Consider $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 > L/|M|$. Then, the unique solution $q = q(t,x)$ of adjoint equation (1.4) with $q_T \in L^2(0,L)$ satisfies*

$$\int_0^L |q(t_1,x)|^2 dx \le \exp\left\{-\frac{3}{4}\left(\frac{1}{44}\right)^{1/3} \frac{(|M|(t_2-t_1)-L)^{4/3}}{(t_2-t_1)^{1/3}\varepsilon^{1/3}}\right\} \int_0^L |q(t_2,x)|^2 dx. \tag{4.1}$$

**Proof.** We adapt the ideas used in [19, Proposition 3.2]. Consider $\phi(t,x) := M(T-t)+x$. For a $\mu \in \mathbb{R}$ that we are going to specify later, multiply equation (1.4) by $\exp\{\mu\phi(t,x)\}q(t,x)$ and then perform integrations by parts to get

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^L e^{\mu\phi}|q|^2 dx\right) + \varepsilon \int_0^L e^{\mu\phi}\left(\mu^2 q + 2\mu q_x + q_{xx}\right) q_{xx}\, dx = 0.$$

Further integrations by parts and the Cauchy inequality lead us to

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\left(\int_0^L e^{\mu\phi}|q|^2 dx\right) + \frac{\varepsilon}{2}\mu^4 \int_0^L e^{\mu\phi}|q|^2 dx + \frac{\varepsilon}{2}\int_0^L e^{\mu\phi}|q_{xx}|^2 dx \\ \le 3\varepsilon\mu^2 \int_0^L e^{\mu\phi}|q_x|^2 dx. \end{aligned} \tag{4.2}$$

We proceed to handle the right-hand side of (4.2). Once again, performing integrations by parts and then using the Cauchy inequality allow us to obtain

$$\begin{aligned} 3\varepsilon\mu^2 \int_0^L e^{\mu\phi}|q_x|^2 dx &= \frac{3\varepsilon}{2}\mu^4 \int_0^L e^{\mu\phi}|q|^2 dx - 3\varepsilon\mu^2 \int_0^L e^{\mu\phi} q q_{xx}\, dx \\ &\le 6\varepsilon\mu^4 \int_0^L e^{\mu\phi}|q|^2 dx + \frac{\varepsilon}{2}\int_0^L e^{\mu\phi}|q_{xx}|^2 dx, \end{aligned}$$

which combined with (4.2) gives us

$$-\frac{d}{dt}\left(e^{-11\varepsilon\mu^4(T-t)}\int_0^L e^{\mu\phi}|q|^2dx\right)\le 0. \tag{4.3}$$

Consider $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 > L/|M|$. Then, integrating (4.3) on $(t_1, t_2)$ yields

$$e^{-11\varepsilon\mu^4(T-t_1)}\int_0^L e^{\mu\phi(t_1,x)}|q(t_1,x)|^2dx \le e^{-11\varepsilon\mu^4(T-t_2)}\int_0^L e^{\mu\phi(t_2,x)}|q(t_2,x)|^2dx. \tag{4.4}$$

Now, we distinguish the following two cases.

**Case 1:** Here we consider $M > 0$ and $\mu > 0$. Taking into account that $\phi(t_2, x) \le M(T - t_2) + L$ and $\phi(t_1, x) \ge M(T - t_1)$ are valid for every $x \in [0, L]$, from (4.4) we get

$$\int_0^L |q(t_1,x)|^2dx \le e^{K_1}\int_0^L |q(t_2,x)|^2dx, \quad K_1 := 11\varepsilon\mu^4(t_2-t_1) - \mu\left(M(t_2-t_1)-L\right). \tag{4.5}$$

Noting that $M(t_2 - t_1) - L > 0$, in (4.5) we choose the $\mu > 0$ that minimizes $K_1$, which is the one given by

$$\mu = \left(\frac{M(t_2-t_1)-L}{44\varepsilon(t_2-t_1)}\right)^{1/3}.$$

Therefore, by plugging this quantity into (4.5) we arrive at (4.1).

**Case 2:** Here we consider $M < 0$ and $\mu < 0$. Taking into account that $\phi(t_2, x) \ge M(T - t_2)$ and $\phi(t_1, x) \le M(T - t_1) + L$ are valid for every $x \in [0, L]$, from (4.4) we get

$$\int_0^L |q(t_1,x)|^2dx \le e^{K_2}\int_0^L |q(t_2,x)|^2dx, \quad K_2 := 11\varepsilon\mu^4(t_2-t_1) + \mu\left(-M(t_2-t_1)-L\right). \tag{4.6}$$

Noting that $-M(t_2 - t_1) - L > 0$, in (4.6) we choose the $\mu < 0$ that minimizes $K_2$, which is the one given by

$$\mu = -\left(\frac{-M(t_2-t_1)-L}{44\varepsilon(t_2-t_1)}\right)^{1/3}.$$

Therefore, by plugging this quantity into (4.6) we arrive at (4.1). The proof of Proposition 4.1 is complete. □

Now we are in position to prove one of our main results.

**Proof of Theorem 1.2.** In virtue of the controllability–observability duality, we need to prove that the unique solution $q = q(t, x)$ of adjoint equation (1.4) with $T \ge 40L/|M|$ and $q_T \in V$ satisfies

$$\int_0^L |q(0,x)|^2 dx \le C_1 \left(\frac{L^3+L^7}{T}\right) \exp\left\{-C_2 \frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\} \times \int_0^T \left(|q_x(t,0)|^2 + |q_{xxx}(t,0)|^2\right) dt. \tag{4.7}$$

Note that the controllability–observability duality also implies an upper bound for the norm of the controls, namely

$$\|u_1\|^2_{L^2(0,T)} + \|u_2\|^2_{L^2(0,T)} \le \frac{C_1}{\varepsilon^2}\left(\frac{L^3+L^7}{T}\right) \exp\left\{-C_2 \frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\} \|z_0\|^2_{L^2(0,L)}.$$

In what follows we focus on the obtention of observability inequality (4.7), where we are going to use that adjoint equation (1.4) has a unique solution $q \in C([0,T];V) \cap L^2(0,T;D(A))$ in virtue of Proposition 2.1 (b).

First, consider any $T > 0$ being such that $2T/3 > L/|M|$. Let $t \in [2T/3, 3T/4]$. In Proposition 4.1 we take $t_1 = 0$ and $t_2 = t$ to obtain

$$\int_0^L |q(0,x)|^2 dx \le \exp\left\{-\frac{3}{4}\left(\frac{1}{44}\right)^{1/3} \frac{(|M|t-L)^{4/3}}{t^{1/3}\varepsilon^{1/3}}\right\} \int_0^L |q(t,x)|^2 dx. \tag{4.8}$$

Since under the previous setting we have

$$\exp\left\{-\frac{3}{4}\left(\frac{1}{44}\right)^{1/3} \frac{(|M|t-L)^{4/3}}{t^{1/3}\varepsilon^{1/3}}\right\} \le \exp\left\{-\frac{3}{4}\left(\frac{1}{33}\right)^{1/3} \frac{\left(\frac{2T}{3}|M|-L\right)^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\},$$

the integration of (4.8) on $(2T/3, 3T/4)$ leads us to

$$\int_0^L |q(0,x)|^2 dx \le \frac{12}{T} \exp\left\{-\frac{3}{4}\left(\frac{1}{33}\right)^{1/3} \frac{\left(\frac{2T}{3}|M|-L\right)^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\} \int_{2T/3}^{3T/4}\int_0^L |q|^2 dx dt. \tag{4.9}$$

Second, in Lemma 3.5 take $n = 0$ and fix

$$s = \frac{9}{40} T^{1/3} L^{-2/3} \varepsilon^{-1/3} + \frac{1}{8} T^{2/3} L^{-1} \varepsilon^{-1/3} |M|^{1/3} + C T^{2/3} L^{-2}.$$

Then, from the combination of (3.4) with (4.9), we get that there exists $C_1 > 0$, independent of $(T, L, \varepsilon) \in (0, +\infty)^3$ and $M \in \mathbb{R} - \{0\}$, such that

$$\int_0^L |q(0,x)|^2 dx \le C_1 \left(\frac{L^3+L^7}{T}\right) \exp\left\{\frac{K(T)}{T^{1/3}\varepsilon^{1/3}}\right\} \int_0^T \left(|q_x(t,0)|^2 + |q_{xxx}(t,0)|^2\right) dt, \tag{4.10}$$

where

$$K(T) := 27 \cdot \frac{9}{40} L^{4/3} + 27 \cdot \frac{1}{8} T^{1/3} L |M|^{1/3} - \frac{3}{4} \left(\frac{1}{33}\right)^{1/3} \left(\frac{2T}{3}|M| - L\right)^{4/3}.$$

Note that the special choice of $T = \tau L/|M|$, with $\tau > 3/2$, gives us

$$K\left(\frac{\tau L}{|M|}\right) = L^{4/3} \left[ 27 \cdot \frac{9}{40} + 27 \cdot \frac{1}{8} \tau^{1/3} - \frac{3}{4} \left(\frac{1}{33}\right)^{1/3} \left(\frac{2\tau}{3} - 1\right)^{4/3} \right]. \tag{4.11}$$

Finally, it can be checked that there exists $\tau* \in (3/2, 40]$ such that $K(\tau L/|M|)$ is a decreasing function on $[\tau^*, +\infty)$. Furthermore, there exists $C_2 > 0$, independent of $(T, L, \varepsilon) \in (0, +\infty)^3$ and $M \in \mathbb{R} - \{0\}$, such that

$$K\left(\frac{40L}{|M|}\right) \leq -C_2 L^{4/3}. \tag{4.12}$$

Since $K(\tau L/|M|) \leq K(40L/|M|)$ when $\tau \geq 40$, we deduce (4.7), and hence (1.6), from (4.10), (4.11) and (4.12). The proof of Theorem 1.2 is complete. □

### 4.2. *Proof of Theorem 1.4*

We adapt the ideas used in [3, Theorem 1.5] and [19, Theorem 1.4]. Let $T < L/|M|$ and set $R := (L - |M|T)/5 > 0$. Throughout the proof, we make distinctions between the cases $M > 0$ and $M < 0$ when needed. Fix a $\widehat{q_T} \in C^\infty([0, L])$ satisfying

$$\left.\begin{array}{l} \text{Supp}(\widehat{q_T}) \subset (L - 2R, L - R) \text{ if } M > 0, \\ \text{Supp}(\widehat{q_T}) \subset (3R, 4R) \text{ if } M < 0. \end{array}\right\} \tag{4.13}$$

Also, we ask it to satisfy

$$\|\widehat{q_T}\|_{L^2(0,L)} = 1. \tag{4.14}$$

Let $\widehat{q} \in C([0, T]; V) \cap L^2(0, T; D(A))$ be the unique solution of adjoint equation (1.4) associated to $q(T, \cdot) = \widehat{q_T}(\cdot)$ given by Proposition 2.1 (b). In view of (4.14), that solution satisfies

$$\int_0^L |\widehat{q}(t, x)|^2 dx + 2\varepsilon \int_t^T \int_0^L |\widehat{q}_{xx}(\tau, x)|^2 dx d\tau = 1, \quad t \in [0, T]. \tag{4.15}$$

Our goal is to prove the following points.

- There exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2} \leq \|\widehat{q}(0, \cdot)\|^2_{L^2(0,L)}, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{4.16}$$

- There exists $C_1 > 0$ such that

$$\|\widehat{q}_{xxx}(\cdot,0)\|^2_{L^2(0,T)} + \|\widehat{q}_x(\cdot,0)\|^2_{L^2(0,T)} \le C_1\left(1+\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon^3}\right)\int_0^T\int_0^R |\widehat{q}|^2 dxdt. \quad (4.17)$$

- There exist $C_1 > 0$ and $C_2 > 0$ such that

$$\int_0^T\int_0^R |\widehat{q}|^2 dxdt \le C_1\left(1+\varepsilon+\frac{1}{\varepsilon}\right)\exp\left\{-\frac{C_2}{T^{1/3}\varepsilon^{1/3}}\right\}. \quad (4.18)$$

Before obtaining (4.16), (4.17) and (4.18), we proceed to explain how these inequalities allow us to conclude the result. For a $z_0 \in L^2(0,L)$ to be selected, we know by Theorem 1.1 that there exists $(u_1,u_2) \in L^2(0,T)^2$ such that the unique solution (defined by transposition) $z \in C([0,T];V^*)$ of equation (1.1) satisfies $z(T,\cdot)=0$ in $V^*$. Hence, multiplying equation (1.1) by $\widehat{q}=\widehat{q}(t,x)$ and then performing some integrations by parts, we arrive at

$$\langle z_0, \widehat{q}(0,\cdot)\rangle_{V^*\times V} = \varepsilon\int_0^T u_1(t)\widehat{q}_{xxx}(t,0)\,dt + \varepsilon\int_0^T u_2(t)\widehat{q}_x(t,0)\,dt.$$

Accordingly, by selecting $z_0(\cdot) := \widehat{q}(0,\cdot) \in V \subset L^2(0,L)$, we get

$$\|\widehat{q}(0,\cdot)\|^2_{L^2(0,L)} \le \frac{\varepsilon}{2}\left(\|u_1\|^2_{L^2(0,T)}+\|u_2\|^2_{L^2(0,T)}\right)\left(\|\widehat{q}_{xxx}(\cdot,0)\|^2_{L^2(0,T)}+\|\widehat{q}_x(\cdot,0)\|^2_{L^2(0,T)}\right),$$

from which we deduce (1.7) thanks to (4.16), (4.17), (4.18) and Young's inequality.

**Proof of (4.16).** Let us introduce $\psi(t,x) := \widehat{q}_T\,(M(T-t)+x)$. Note that $\psi(t,\cdot) \in C_0^\infty([0,L])$ for every $t \in [0,T]$ according to (4.13). Hence, multiplying adjoint equation (1.4) by $\psi = \psi(t,x)$, then performing some integrations by parts and finally using (4.14), we obtain

$$\begin{aligned}1 &= \int_0^L \widehat{q}(0,x)\psi(0,x)\,dx + \varepsilon\int_0^T\int_0^L \widehat{q}\psi_{xxxx}\,dxdt \\ &\le \frac{1}{2}\|\widehat{q}(0,\cdot)\|^2_{L^2(0,L)} + \frac{1}{2}\|\psi(0,\cdot)\|^2_{L^2(0,L)} + \varepsilon\|\widehat{q}\|_{L^2(0,T;L^2(0,L))}\|\psi_{xxxx}\|_{L^2(0,T;L^2(0,L))}.\end{aligned} \quad (4.19)$$

We proceed to handle the right-hand side of (4.19). On the one hand, (4.13), the change of variable $MT+x \to x$ and (4.14) allow us to deduce

$$\int_0^L |\psi(0,x)|^2 dx = \int_{\mathrm{Supp}(\widehat{q_T})} |\widehat{q}_T(x)|^2 dx = 1. \quad (4.20)$$

On the other hand, (4.13) and the change of variables $(t, M(T-t)+x) \to (t, x)$ imply

$$\int_0^T \int_0^L |\psi_{xxxx}|^2 dx dt = \int_0^T \int_{\mathrm{Supp}(\widehat{q_T})} |\widehat{q}_T''''(x)|^2 dx dt = T\|\widehat{q}_T''''\|^2_{L^2(0,L)}. \tag{4.21}$$

Therefore, since $\|\widehat{q}\|_{L^2(0,T;L^2(0,L))} \leq T^{1/2}$ is valid due to (4.15), we see that the combination of (4.19) together with (4.20) and (4.21) lead us to

$$\left(\frac{1}{2} - \varepsilon T\|\widehat{q}_T''''\|_{L^2(0,L)}\right) \leq \frac{1}{2}\|\widehat{q}(0,\cdot)\|^2_{L^2(0,L)}, \tag{4.22}$$

from which we deduce (4.16) by setting

$$\varepsilon_0 := \frac{1}{4T\|\widehat{q}_T''''\|_{L^2(0,L)}}. \quad \square$$

**Proof of (4.17).** First, multiplying adjoint equation (1.4) by $(R-x)^4\widehat{q} = (R-x)^4\widehat{q}(t,x)$ and then performing some integrations by parts, we arrive at

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^R (R-x)^4|\widehat{q}|^2 dx\right) + \varepsilon\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx + \varepsilon\int_0^R (R-x)^2|\widehat{q}_x|^2 dx + 12\varepsilon\int_0^R |\widehat{q}|^2 dx$$
$$+ 4\varepsilon R^3|\widehat{q}_x(t,0)|^2 = 2M\int_0^R (R-x)^3|\widehat{q}|^2 dx + 25\varepsilon\int_0^R (R-x)^2|\widehat{q}_x|^2 dx. \tag{4.23}$$

In order to deal with the last term of the right-hand side of (4.23), we perform one integration by parts and then employ the Cauchy inequality to get

$$25\varepsilon\int_0^R (R-x)^2|\widehat{q}_x|^2 dx \leq C\varepsilon\int_0^R |\widehat{q}|^2 dx + \frac{\varepsilon}{2}\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx.$$

Hence, plugging this inequality into (4.23), then integrating the resulting expression on $(0, T)$ and finally using the fact that $\widehat{q_T}(x) = 0$ for $x \in [0, R]$, which is a consequence of (4.13), give us

$$\int_0^R (R-x)^4|\widehat{q}(0,x)|^2 dx + \varepsilon\int_0^T\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx dt + \varepsilon\int_0^T\int_0^R (R-x)^2|\widehat{q}_x|^2 dx dt$$
$$+ 8\varepsilon R^3\|\widehat{q}_x(\cdot,0)\|^2_{L^2(0,T)} \leq C(\varepsilon + |M|R^3)\int_0^T\int_0^R |\widehat{q}|^2 dx dt. \tag{4.24}$$

Second, multiplying adjoint equation (1.4) by $(R-x)^8\widehat{q}_{xxxx} = (R-x)^8\widehat{q}_{xxxx}(t,x)$ and then performing some integrations by parts, we arrive at

$$-\frac{1}{2}\frac{d}{dt}\left(\int_0^R (R-x)^8|\widehat{q}_{xx}|^2 dx\right) + \varepsilon\int_0^R (R-x)^8|\widehat{q}_{xxxx}|^2 dx$$
$$= 16\int_0^R (R-x)^7\widehat{q}_t\widehat{q}_{xxx}\,dx - 56\int_0^R (R-x)^6\widehat{q}_t\widehat{q}_{xx}\,dx + M\int_0^R (R-x)^8\widehat{q}_x\widehat{q}_{xxxx}\,dx. \quad (4.25)$$

We proceed to handle the right-hand side of (4.25), and in order to do so, we are going to use that $\widehat{q}_t = \varepsilon\widehat{q}_{xxxx} - M\widehat{q}_x$. For the first term, we perform integration by parts twice and then employ the Cauchy inequality to obtain

$$16\int_0^R (R-x)^7\widehat{q}_t\widehat{q}_{xxx}\,dx \le 56\varepsilon\int_0^R (R-x)^6|\widehat{q}_{xxx}|^2 - 8\varepsilon R^7|\widehat{q}_{xxx}(t,0)|^2$$
$$+ C|M|R^3\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx + C|M|R^3\int_0^R (R-x)^2|\widehat{q}_x|^2 dx. \quad (4.26)$$

As we did previously, we have

$$56\varepsilon\int_0^R (R-x)^6|\widehat{q}_{xxx}|^2 dx \le C\varepsilon\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx + \frac{\varepsilon}{6}\int_0^R (R-x)^8|\widehat{q}_{xxxx}|^2 dx,$$

which combined with (4.26) yields

$$16\int_0^R (R-x)^7\widehat{q}_t\widehat{q}_{xxx}\,dx \le -8\varepsilon R^7|\widehat{q}_{xxx}(t,0)|^2 + C|M|R^3\int_0^R (R-x)^2|\widehat{q}_x|^2$$
$$+ C\left(\varepsilon + |M|R^3\right)\int_0^R (R-x)^4|\widehat{q}_{xx}|^2 dx + \frac{\varepsilon}{6}\int_0^R (R-x)^8|\widehat{q}_{xxxx}|^2 dx. \quad (4.27)$$

To handle the remaining two terms, we employ the Cauchy inequality to get

$$-56\int_0^R (R-x)^6\widehat{q}_t\widehat{q}_{xx}\,dx + M\int_0^R (R-x)^8\widehat{q}_x\widehat{q}_{xxxx}\,dx \le C\frac{|M|^2R^6}{\varepsilon}\int_0^R (R-x)^2|\widehat{q}_x|^2$$

$$+ C\varepsilon \int_0^R (R-x)^4 |\widehat{q}_{xx}|^2 dx + \frac{\varepsilon}{3} \int_0^R (R-x)^8 |\widehat{q}_{xxxx}|^2 dx. \quad (4.28)$$

Hence, plugging (4.27) and (4.28) into (4.25), then integrating the resulting expression on $(0, T)$ and finally using the fact that $\widehat{q}_T''(x) = 0$ for $x \in [0, R]$, which is a consequence of (4.13), give us

$$\int_0^R (R-x)^8 |\widehat{q}_{xx}(0,x)|^2 dx + \varepsilon \int_0^T \int_0^R (R-x)^8 |\widehat{q}_{xxxx}|^2 dx dt$$
$$+ 16\varepsilon R^7 \|\widehat{q}_{xxx}(\cdot, 0)\|^2_{L^2(0,T)} \le C \left( |M| R^3 + \frac{|M|^2 R^6}{\varepsilon} \right) \int_0^T \int_0^R (R-x)^2 |\widehat{q}_x|^2 dx dt$$
$$+ C \left( \varepsilon + |M| R^3 \right) \int_0^T \int_0^R (R-x)^4 |\widehat{q}_{xx}|^2 dx dt.$$

Finally, from the combination of (4.24) with the previous inequality we deduce

$$\|\widehat{q}_{xxx}(\cdot, 0)\|^2_{L^2(0,T)} + \|\widehat{q}_x(\cdot, 0)\|^2_{L^2(0,T)}$$
$$\le C \left( \frac{1}{R^3} + \frac{1}{R^7} \right) \left( 1 + \frac{|M| R^3}{\varepsilon} + \frac{|M|^2 R^6}{\varepsilon^2} + \frac{|M|^3 R^9}{\varepsilon^3} \right) \int_0^T \int_0^R |\widehat{q}|^2 dx dt,$$

which leads us to (4.17). □

**Proof of (4.18).** Let us fix a $\varphi \in C^\infty(\mathbb{R})$ such that

$$\varphi \ge 0 \text{ and } \varphi' \le 0 \text{ in } \mathbb{R}. \quad (4.29)$$

Also, we ask it to satisfy

$$\left.\begin{array}{l} \varphi = 1 \text{ in } (-\infty, L-3R] \text{ and } \varphi = 0 \text{ in } [L-2R, +\infty) \text{ if } M > 0, \\ \varphi = 1 \text{ in } (-\infty, 2R] \text{ and } \varphi = 0 \text{ in } [3R, +\infty) \text{ if } M < 0. \end{array}\right\} \quad (4.30)$$

Following the idea used in the proof of Proposition 4.1, we define $\phi(t, x) := M(T-t) + x$. For a $\mu > 0$ that we are going to specify later, multiply adjoint equation (1.4) by $\varphi(\phi) \exp(-\mu\phi) \widehat{q}$ and then perform several integrations by parts to get

$$-\frac{1}{2} \frac{d}{dt} \left( \int_0^L \varphi(\phi) e^{-\mu\phi} |\widehat{q}|^2 dx \right) + \varepsilon \int_0^L \varphi(\phi) e^{-\mu\phi} |\widehat{q}_{xx}|^2 dx + \varepsilon \left( \varphi(\phi) e^{-\mu\phi} \right)_x |\widehat{q}_x|^2 \Big|_{x=0}^{x=L}$$

$$= \frac{\varepsilon}{2}\int_0^L \left(\varphi(\phi)e^{-\mu\phi}\right)_{xxxx}|\widehat{q}|^2 dx - 2\varepsilon\int_0^L \left(\varphi(\phi)e^{-\mu\phi}\right)_{xx}\widehat{q}_{xx}\widehat{q}\,dx. \quad (4.31)$$

We proceed to handle some terms of (4.31). For the boundary terms, we have that (4.29) and (4.30), the latter implying $\varphi(\phi(t,L)) = \varphi'(\phi(t,L)) = 0$ for $t \in [0,T]$ because of

$$\left.\begin{array}{l}\phi(t,L) \geq L \text{ if } M > 0,\\ \phi(t,L) \geq 5R \text{ if } M < 0,\end{array}\right\}$$

allow us to deduce

$$\left(\varphi(\phi)e^{-\mu\phi}\right)_x |\widehat{q}_x|^2\Big|_{x=0}^{x=L} = -\left[\varphi'(\phi(t,0)) - \mu\varphi(\phi(t,0))\right]e^{-\mu\phi(t,0)}|\widehat{q}_x(t,0)|^2 \geq 0. \quad (4.32)$$

For the first term of the right-hand side of (4.31), we have

$$\begin{aligned}\frac{\varepsilon}{2}\int_0^L \left(\varphi(\phi)e^{-\mu\phi}\right)_{xxxx}|\widehat{q}|^2 dx &\leq \frac{\varepsilon\mu^4}{2}\int_0^L \varphi(\phi)e^{-\mu\phi}|\widehat{q}|^2 dx\\ &+ C\varepsilon(1+\mu+\mu^2+\mu^3)\int_0^L \left(|\varphi'(\phi)| + |\varphi''(\phi)| + |\varphi'''(\phi)| + |\varphi''''(\phi)|\right)e^{-\mu\phi}|\widehat{q}|^2 dx. \quad (4.33)\end{aligned}$$

Let us introduce

$$\left.\begin{array}{l}I(t) = [L-3R-M(T-t), L-2R-M(T-t)] \text{ if } M > 0,\\ I(t) = [2R-M(T-t), 3R-M(T-t)] \text{ if } M < 0.\end{array}\right\} \quad (4.34)$$

Then, (4.30), (4.34) and the change of variable $M(T-t)+x \to x$ imply

$$\int_0^L \left(|\varphi'(\phi)| + |\varphi''(\phi)| + |\varphi'''(\phi)| + |\varphi''''(\phi)|\right)e^{-\mu\phi}|\widehat{q}|^2 dx \leq \|\varphi\|_{W^{4,\infty}(I(T))}\int_{I(t)} e^{-\mu\phi}|\widehat{q}|^2 dx,$$

which combined with (4.33) yields

$$\begin{aligned}\frac{\varepsilon}{2}\int_0^L \left(\varphi(\phi)e^{-\mu\phi}\right)_{xxxx}|\widehat{q}|^2 dx &\leq \frac{\varepsilon\mu^4}{2}\int_0^L \varphi(\phi)e^{-\mu\phi}|\widehat{q}|^2 dx\\ &+ C\varepsilon(1+\mu+\mu^2+\mu^3)\int_{I(t)} e^{-\mu\phi}|\widehat{q}|^2 dx. \quad (4.35)\end{aligned}$$

For the second term of the right-hand side of (4.31), we proceed as we just did for the obtention of (4.35) and then employ the Cauchy inequality to obtain

$$-2\varepsilon\int_0^L \left(\varphi(\phi)e^{-\mu\phi}\right)_{xx}\widehat{q}_{xx}\widehat{q}\,dx \le \varepsilon\int_0^L \varphi(\phi)e^{-\mu\phi}|\widehat{q}_{xx}|^2dx + \varepsilon\mu^4\int_0^L \varphi(\phi)e^{-\mu\phi}|\widehat{q}|^2dx$$
$$+ C\varepsilon(1+\mu)\int_{I(t)} e^{-\mu\phi}(|\widehat{q}|^2+|\widehat{q}_{xx}|^2)dx. \quad (4.36)$$

Accordingly, from the combination of (4.31), (4.32), (4.35) and (4.36), we arrive at

$$-\frac{d}{dt}\left(e^{-3\varepsilon\mu^4(T-t)}\int_0^L \varphi(\phi)e^{-\mu\phi}|\widehat{q}|^2dx\right)$$
$$\le C\varepsilon(1+\mu+\mu^2+\mu^3)e^{-3\varepsilon\mu^4(T-t)}\int_{I(t)} e^{-\mu\phi}\left(|\widehat{q}|^2+|\widehat{q}_{xx}|^2\right)dx. \quad (4.37)$$

Let us introduce

$$\alpha := \begin{cases} L-3R & , \quad \text{if } M>0, \\ 2R & , \quad \text{if } M<0, \end{cases}$$

and note that from (4.34) we have $\alpha \le \phi(t,x)$ for $(t,x)\times[0,T]\times I(t)$. Taking into account that $\text{Supp}(\widehat{q}_T)\cap\text{Supp}(\varphi)=\emptyset$ is valid due to (4.13) and (4.30), integrating (4.37) on $(t,T)$ and then using (4.15) lead us to

$$\int_0^L \varphi(\phi(t,x))e^{-\mu\phi(t,x)}|\widehat{q}(t,x)|^2dx \le C(1+\mu+\mu^2+\mu^3)e^{3\varepsilon\mu^4T-\mu\alpha}(\varepsilon T+1). \quad (4.38)$$

Now, we truncate the integral of the left-hand side of (4.38). Introducing

$$\beta := \begin{cases} L-4R & , \quad \text{if } M>0, \\ R & , \quad \text{if } M<0, \end{cases}$$

we see that $\phi(t,x)\le\beta$ for $(t,x)\in[0,T]\times[0,R]$, implying $\varphi(\phi)=1$ in $[0,T]\times[0,R]$ in virtue of (4.30). Hence, using $\beta-\alpha=-R$ in (4.38) gives us

$$\int_0^R |\widehat{q}(t,x)|^2dx \le C(1+\mu+\mu^2+\mu^3)e^{K}(\varepsilon T+1), \quad K := 3\varepsilon\mu^4T - R\mu. \quad (4.39)$$

Finally, we choose the $\mu>0$ that minimizes $K$, which is the one given by

$$\mu = \frac{R^{1/3}}{12^{1/3}T^{1/3}\varepsilon^{1/3}}.$$

Therefore, plugging this quantity into (4.39) we deduce (4.18) in view of Young's inequality. The proof of Theorem 1.4 is complete. □

## 5. Further remarks

In this section, we extend our main results to other boundary conditions in equation (1.1). To this end, let us consider the fourth-order parabolic equation

$$\begin{cases} z_t + \varepsilon z_{xxxx} + M z_x = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = u_1(t),\ z(t,L) = 0, & t \in (0,T), \\ z_x(t,0) = u_2(t),\ z_x(t,L) = 0, & t \in (0,T), \\ z(0,x) = z_0(x), & x \in (0,L). \end{cases} \tag{5.1}$$

In this case, the corresponding adjoint equation is given by

$$\begin{cases} -q_t + \varepsilon q_{xxxx} - M q_x = 0, & (t,x) \in (0,T) \times (0,L), \\ q(t,0) = 0,\ q(t,L) = 0, & t \in (0,T), \\ q_x(t,0) = 0,\ q_x(t,L) = 0, & t \in (0,T), \\ q(T,x) = q_T(x), & x \in (0,L). \end{cases} \tag{5.2}$$

Here we consider $V := H_0^2(0,L)$ and keep identifying $L^2(0,L)$ with itself, obtaining $V^* = H^{-2}(0,L)$. Note that all the well-posedness results needed for studying equation (5.1) can be adapted from [5, Section 2].

- **Null controllability.** As in equation (1.1), the null controllability of equation (5.1) with only one boundary control is an open problem (see Remark 1.6). Thanks to Remark 3.4, Theorem 1.1 and Corollary 1.3 remain valid for equation (5.1). Note that we could use the Carleman estimates [5, Theorem 3.5] or [20, Proposition 2] to deduce the null controllability part of Theorem 1.1. However, these estimates do not allow us to get (1.3) because of the non-optimality of their weight functions (see Remark 3.3).
- **Cost of null controllability.** From the previous point, we have that the cost of null controllability of equation (5.1), defined in (1.2), is well defined if we take $Z$ as $V^*$, or any subspace of it. In this case, the exponential dissipation result for adjoint equation (5.2) is better than the one for adjoint equation (1.4).

**Proposition 5.1.** *Let $M \in \mathbb{R} - \{0\}$. Consider $0 \le t_1 < t_2 \le T$ such that $t_2 - t_1 > L/|M|$. Then, the unique solution $q = q(t,x)$ of adjoint equation (5.2) with $q_T \in L^2(0,L)$ satisfies*

$$\int_0^L |q(t_1,x)|^2 dx \le \exp\left\{ -\left(\frac{3}{16}\right)^{2/3} \frac{(|M|(t_2-t_1)-L)^{4/3}}{(t_2-t_1)^{1/3}\varepsilon^{1/3}} \right\} \int_0^L |q(t_2,x)|^2 dx. \tag{5.3}$$

The slightly improvement from $-\frac{3}{4}\left(\frac{1}{44}\right)^{1/3}$ in (4.1) to $-\left(\frac{3}{16}\right)^{2/3}$ in (5.3) is due to the boundary conditions of adjoint equation (5.2) and allows us to get a lower control time for the uniform null controllability, with respect to the diffusion coefficient, of equation (5.1) when the initial condition is in $L^2(0,L)$.

**Theorem 5.2.** *Let $T \ge 28L/|M|$ with $M \in \mathbb{R} - \{0\}$. For every $z_0 \in L^2(0,L)$, there exists $(u_1,u_2) \in L^2(0,T)^2$ such that the unique solution (defined by transposition) $z \in C([0,T]; V^*)$*

*of equation (1.1) satisfies $z(T,\cdot)=0$ in $V^*$. Moreover, there exist $C_1>0$ and $C_2>0$, both independent of $(T,L,\varepsilon)\in(0,+\infty)^3$ and $M\in\mathbb{R}-\{0\}$, such that*

$$\|u_1\|^2_{L^2(0,T)}+\|u_2\|^2_{L^2(0,T)}\le\frac{C_1}{\varepsilon^2}\left(\frac{L^3+L^7}{T}\right)\exp\left\{-C_2\frac{L^{4/3}}{T^{1/3}\varepsilon^{1/3}}\right\}\|z_0\|^2_{L^2(0,L)}. \tag{5.4}$$

As in the proof of Theorem 1.2, the main tools for proving Theorem 5.2 are the exponential dissipation result given by Proposition 5.1 and the Carleman estimate given by Proposition 3.1 (see Remark 3.4). Finally, Theorem 1.4 and Corollary 1.5 remain valid for equation (5.1).

## Acknowledgments

We would like to thank Eduardo Cerpa for presenting us this interesting problem. This work has been partially supported by FONDECYT 3150089 (N. Carreño), FONDECYT 1140741 (P. Guzmán), CONICYT grant ACT 1106, Math-Amsud Project COSIP 14MATH-03 and Basal Project FB0008.

## Appendix A. Carleman estimate

In this appendix, we prove the Carleman estimate given by Proposition 3.1. Recall the notations introduced at the beginning of Section 3.

**Proof of Proposition 3.1.** Let us set $\mathcal{L}q:=-q_t+\varepsilon q_{xxxx}-Mq_x$. For $s>0$, let us consider $w:=e^{-s\alpha}q$, $\mathcal{P}w:=e^{-s\alpha}\mathcal{L}(e^{s\alpha}w)$ and the decomposition $\mathcal{P}w=\mathcal{P}_1w+\mathcal{P}_2w+\mathcal{P}_3w$ given by

$$\begin{aligned}
\mathcal{P}_1w &:= -w_t+4\varepsilon s^3\alpha_x^3w_x+4\varepsilon s\alpha_xw_{xxx}+3\varepsilon s^3\alpha_x\left(\alpha_x^2\right)_x w-Mw_x,\\
\mathcal{P}_2w &:= \left(\varepsilon s^4\alpha_x^4-s\alpha_t\right)w+6\varepsilon s^2\alpha_x^2w_{xx}+\varepsilon w_{xxxx}+6\varepsilon s^2\left(\alpha_x^2\right)_x w_x-Ms\alpha_xw,\\
\mathcal{P}_3w &:= 3\varepsilon s^2\alpha_{xx}^2w+6\varepsilon s\alpha_{xx}w_{xx}.
\end{aligned} \tag{A.1}$$

Independent of the decomposition that we could choose for $\mathcal{P}w$, taking the $L^2$-norm in $Q:=(0,T)\times(0,L)$ to $\mathcal{P}_1w+\mathcal{P}_2w=\mathcal{P}w-\mathcal{P}_3w$, we get

$$\|\mathcal{P}_1w\|^2_{L^2(Q)}+2\left(\mathcal{P}_1w,\mathcal{P}_2w\right)_{L^2(Q)}+\|\mathcal{P}_2w\|^2_{L^2(Q)}=\|\mathcal{P}w-\mathcal{P}_3w\|^2_{L^2(Q)},$$

from which it follows

$$(\mathcal{P}_1w,\mathcal{P}_2w)_{L^2(Q)}\le\|\mathcal{P}w\|^2_{L^2(Q)}+\|\mathcal{P}_3w\|^2_{L^2(Q)}. \tag{A.2}$$

The key point of the proof is to careful manipulate the terms that will appear when computing $(\mathcal{P}_1w,\mathcal{P}_2w)_{L^2(Q)}$.

**Computing** $(\mathcal{P}_1w,\mathcal{P}_2w)_{L^2(Q)}$. For $(i,j)\in\{1,2,3,4,5\}^2$, we denote by $I_{i,j}$ the $L^2$-product in $Q$ between the $i$-th term of $\mathcal{P}_1w$ with the $j$-th term of $\mathcal{P}_2w$. Integrations by parts are performed and each resulting expression for $I_{i,j}$ is listed below.

- $I_{1,1} = 2\varepsilon s^4 \iint\limits_Q \alpha_x^3 \alpha_{xt} |w|^2 dxdt - \frac{s}{2} \iint\limits_Q \alpha_{tt} |w|^2 dxdt.$
- $I_{1,2} = 12\varepsilon s^2 \iint\limits_Q \alpha_x \alpha_{xx} w_t w_x \, dxdt - 6\varepsilon s^2 \iint\limits_Q \alpha_x \alpha_{xt} |w_x|^2 dxdt.$
- $I_{1,3} = \varepsilon \int_0^T w_{xt} w_{xx} |_{x=0}^{x=L} \, dt.$
- $I_{1,4} = -12\varepsilon s^2 \iint\limits_Q \alpha_x \alpha_{xx} w_t w_x \, dxdt.$
- $I_{1,5} = -\frac{sM}{2} \iint\limits_Q \alpha_{xt} |w|^2 dxdt.$
- $$I_{2,1} = -14\varepsilon^2 s^7 \iint\limits_Q \alpha_x^6 \alpha_{xx} |w|^2 dxdt + 6\varepsilon s^4 \iint\limits_Q \alpha_x^2 \alpha_{xx} \alpha_t |w|^2 dxdt + 2\varepsilon s^4 \iint\limits_Q \alpha_x^3 \alpha_{xt} |w|^2 dxdt.$$
- $I_{2,2} = -60\varepsilon^2 s^5 \iint\limits_Q \alpha_x^4 \alpha_{xx} |w_x|^2 dxdt + 12\varepsilon^2 s^5 \int_0^T \alpha_x^5 |w_x|^2 \Big|_{x=0}^{x=L} dt.$
- $$\begin{aligned} I_{2,3} = &-12\varepsilon^2 s^3 \iint\limits_Q \alpha_{xx}^3 |w_x|^2 dxdt + 18\varepsilon^2 s^3 \iint\limits_Q \alpha_x^2 \alpha_{xx} |w_{xx}|^2 dxdt \\ &+ 12\varepsilon^2 s^3 \int_0^T \alpha_x \alpha_{xx}^2 |w_x|^2 \Big|_{x=0}^{x=L} dt \\ &- 2\varepsilon^2 s^3 \int_0^T \alpha_x^3 |w_{xx}|^2 \Big|_{x=0}^{x=L} dt - 12\varepsilon^2 s^3 \int_0^T \alpha_x^2 \alpha_{xx} w_x w_{xx} \Big|_{x=0}^{x=L} dt \\ &+ 4\varepsilon^2 s^3 \int_0^T \alpha_x^3 w_x w_{xxx} \Big|_{x=0}^{x=L} dt. \end{aligned}$$
- $I_{2,4} = 48\varepsilon^2 s^5 \iint\limits_Q \alpha_x^4 \alpha_{xx} |w_x|^2 dxdt.$
- $I_{2,5} = 8\varepsilon s^4 M \iint\limits_Q \alpha_x^3 \alpha_{xx} |w|^2 dxdt.$

- $$I_{3,1} = -120\varepsilon^2 s^5 \iint_Q \alpha_x^2 \alpha_{xx}^3 |w|^2 dxdt + 6\varepsilon s^2 \iint_Q \alpha_{xxt}\alpha_{xx}|w|^2 dxdt + 30\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx}|w_x|^2 dxdt - 6\varepsilon s^2 \iint_Q \alpha_x \alpha_{xt}|w_x|^2 dxdt - 6\varepsilon s^2 \iint_Q \alpha_t \alpha_{xx}|w_x|^2 dxdt - 2\varepsilon^2 s^5 \int_0^T \alpha_x^5 |w_x|^2 \Big|_{x=0}^{x=L} dt + 2\varepsilon s^2 \int_0^T \alpha_t \alpha_x |w_x|^2 \Big|_{x=0}^{x=L} dt.$$
- $$I_{3,2} = -36\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx}|w_{xx}|^2 dxdt + 12\varepsilon^2 s^3 \int_0^T \alpha_x^3 |w_{xx}|^2 \Big|_{x=0}^{x=L} dt.$$
- $$I_{3,3} = -2\varepsilon^2 s \iint_Q \alpha_{xx}|w_{xxx}|^2 dxdt + 2\varepsilon^2 s \int_0^T \alpha_x |w_{xxx}|^2 \Big|_{x=0}^{x=L} dt.$$
- $$I_{3,4} = 48\varepsilon^2 s^3 \iint_Q \alpha_{xx}^3 |w_x|^2 dxdt - 48\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx}|w_{xx}|^2 dxdt + 48\varepsilon^2 s^3 \int_0^T \alpha_x^2 \alpha_{xx} w_x w_{xx} \Big|_{x=0}^{x=L} dt - 48\varepsilon^2 s^3 \int_0^T \alpha_x \alpha_{xx}^2 |w_x|^2 \Big|_{x=0}^{x=L} dt.$$
- $$I_{3,5} = -12\varepsilon s^2 M \iint_Q \alpha_x \alpha_{xx}|w_x|^2 dxdt + 2\varepsilon s^2 M \int_0^T \alpha_x^2 |w_x|^2 \Big|_{x=0}^{x=L} dt.$$
- $$I_{4,1} = 6\varepsilon^2 s^7 \iint_Q \alpha_x^6 \alpha_{xx}|w|^2 dxdt - 6\varepsilon s^4 \iint_Q \alpha_x^2 \alpha_{xx}\alpha_t |w|^2 dxdt.$$
- $$I_{4,2} = 216\varepsilon^2 s^5 \iint_Q \alpha_x^2 \alpha_{xx}^3 |w|^2 dxdt - 36\varepsilon^2 s^5 \iint_Q \alpha_x^4 \alpha_{xx}|w_x|^2 dxdt.$$
- $$I_{4,3} = -24\varepsilon^2 s^3 \iint_Q \alpha_{xx}^3 |w_x|^2 dxdt + 6\varepsilon^2 s^3 \iint_Q \alpha_x^2 \alpha_{xx}|w_{xx}|^2 dxdt + 12\varepsilon^2 s^3 \int_0^T \alpha_x \alpha_{xx}^2 |w_x|^2 \Big|_{x=0}^{x=L} dt - 6\varepsilon^2 s^3 \int_0^T \alpha_x^2 \alpha_{xx} w_x w_{xx} \Big|_{x=0}^{x=L} dt.$$
- $$I_{4,4} = -108\varepsilon^2 s^5 \iint_Q \alpha_x^2 \alpha_{xx}^3 |w|^2 dxdt.$$
- $$I_{4,5} = -6\varepsilon s^4 M \iint_Q \alpha_x^3 \alpha_{xx}|w|^2 dxdt.$$

- $I_{5,1} = 2\varepsilon s^4 M \iint\limits_Q \alpha_x^3 \alpha_{xx} |w|^2 dxdt - \dfrac{sM}{2} \iint\limits_Q \alpha_{xt} |w|^2 dxdt.$

- $I_{5,2} = 6\varepsilon s^2 M \iint\limits_Q \alpha_x \alpha_{xx} |w_x|^2 dxdt - 3\varepsilon s^2 M \int_0^T \alpha_x^2 |w_x|^2 \Big|_{x=0}^{x=L} dt.$

- $I_{5,3} = -\varepsilon M \int_0^T w_x w_{xxx}|_{x=0}^{x=L} dt + \dfrac{\varepsilon M}{2} \int_0^T |w_{xx}|^2 \Big|_{x=0}^{x=L} dt.$

- $I_{5,4} = -12\varepsilon s^2 M \iint\limits_Q \alpha_x \alpha_{xx} |w_x|^2 dxdt.$

- $I_{5,5} = -\dfrac{sM^2}{2} \iint\limits_Q \alpha_{xx} |w|^2 dxdt.$

Taking into account the previous computations, we define the distributed and boundary terms respectively by

$$
\begin{aligned}
\mathcal{D}(w) := &-8\varepsilon^2 s^7 \iint\limits_Q \alpha_x^6 \alpha_{xx} |w|^2 dxdt - 18\varepsilon^2 s^5 \iint\limits_Q \alpha_x^4 \alpha_{xx} |w_x|^2 dxdt \\
&- 60\varepsilon^2 s^3 \iint\limits_Q \alpha_x^2 \alpha_{xx} |w_{xx}|^2 dxdt - 2\varepsilon^2 s \iint\limits_Q \alpha_{xx} |w_{xxx}|^2 dxdt \\
&+ 4\varepsilon s^4 \iint\limits_Q \alpha_x^3 \alpha_{xt} |w|^2 dxdt - \frac{s}{2} \iint\limits_Q \alpha_{tt} |w|^2 dxdt - 12\varepsilon s^2 \iint\limits_Q \alpha_x \alpha_{xt} |w_x|^2 dxdt \\
&- sM \iint\limits_Q \alpha_{xt} |w|^2 dxdt + 12\varepsilon^2 s^3 \iint\limits_Q \alpha_{xx}^3 |w_x|^2 dxdt + 4\varepsilon s^4 M \iint\limits_Q \alpha_x^3 \alpha_{xx} |w|^2 dxdt \\
&- 12\varepsilon^2 s^5 \iint\limits_Q \alpha_x^2 \alpha_{xx}^3 |w|^2 dxdt + 6\varepsilon s^2 \iint\limits_Q \alpha_{xxt} \alpha_{xx} |w|^2 dxdt \\
&- 6\varepsilon s^2 \iint\limits_Q \alpha_t \alpha_{xx} |w_x|^2 dxdt \\
&- 18\varepsilon s^2 M \iint\limits_Q \alpha_x \alpha_{xx} |w_x|^2 dxdt - \frac{sM^2}{2} \iint\limits_Q \alpha_{xx} |w|^2 dxdt, \qquad \text{(A.3)}
\end{aligned}
$$

$$
\mathcal{B}(w,x) := 10\varepsilon^2 s^5 \int_0^T \alpha_x^5 |w_x|^2 dt + 10\varepsilon^2 s^3 \int_0^T \alpha_x^3 |w_{xx}|^2 dt + 2\varepsilon^2 s \int_0^T \alpha_x |w_{xxx}|^2 dt
$$

$$+\varepsilon\int_0^T w_{xt}w_{xx}\,dt - 24\varepsilon^2 s^3\int_0^T \alpha_x\alpha_{xx}^2|w_x|^2dt + 30\varepsilon^2 s^3\int_0^T \alpha_x^2\alpha_{xx}w_x w_{xx}\,dt$$

$$+4\varepsilon^2 s^3\int_0^T \alpha_x^3 w_x w_{xxx}\,dt + 2\varepsilon s^2\int_0^T \alpha_t\alpha_x|w_x|^2dt - \varepsilon s^2 M\int_0^T \alpha_x^2|w_x|^2dt$$

$$-\varepsilon M\int_0^T w_x w_{xxx}\,dt + \frac{\varepsilon M}{2}\int_0^T |w_{xx}|^2dt, \quad x\in\{0,L\}. \tag{A.4}$$

Accordingly, the previous computations together with (A.3) and (A.4) lead us to

$$(\mathcal{P}_1 w, \mathcal{P}_2 w)_{L^2(Q)} = \mathcal{D}(w) + \mathcal{B}(w,L) - \mathcal{B}(w,0). \tag{A.5}$$

The next two parts of the proof are devoted to handle the terms in (A.3) and (A.4). To do this, we are going to use some properties of the weight functions defined in (3.1) that can be deduced from

$$1 \le \frac{T^{2/3}}{4^{1/3}}\beta(t), \quad |\beta'(t)| \le \frac{T}{3}\beta(t)^4, \quad |\beta''(t)| \le \frac{11T^2}{18}\beta(t)^7, \quad t\in[0,T], \tag{A.6}$$

$$\frac{L^2}{2} \le -\frac{1}{2}x^2 + 8Lx + \frac{L^2}{2} \le 8L^2, \quad 7L \le \frac{d}{dx}\left(-\frac{1}{2}x^2 + 8Lx + \frac{L^2}{2}\right) \le 8L, \quad x\in[0,L]. \tag{A.7}$$

**Handling** $\mathcal{D}(w)$. We first handle the first four terms in (A.3), which are the leading terms among the distributed terms. Using $\alpha_{xx}(t,x) = -\beta(t)$ and $\alpha_x(t,x) \ge 7L\beta(t)$ into them yields

$$-8\varepsilon^2 s^7\iint_Q \alpha_x^6\alpha_{xx}|w|^2dxdt - 18\varepsilon^2 s^5\iint_Q \alpha_x^4\alpha_{xx}|w_x|^2dxdt - 60\varepsilon^2 s^3\iint_Q \alpha_x^2\alpha_{xx}|w_{xx}|^2dxdt$$

$$-2\varepsilon^2 s\iint_Q \alpha_{xx}|w_{xxx}|^2dxdt \ge 8\cdot 7^6\varepsilon^2\iint_Q s^7L^6\beta^7|w|^2dxdt + 18\cdot 7^4\varepsilon^2\iint_Q s^5L^4\beta^5|w_x|^2dxdt$$

$$+60\cdot 7^2\varepsilon^2\iint_Q s^3L^2\beta^3|w_{xx}|^2dxdt + 2\varepsilon^2\iint_Q s\beta|w_{xxx}|^2dxdt. \tag{A.8}$$

The terms in (A.8) will allow us to absorb the remaining terms of (A.3) by choosing $s>0$ in a suitable way. The following inequalities can be obtained by using the weight function defined in (3.1) together with (A.6) and (A.7).

- $$\left|4\varepsilon s^4\iint_Q \alpha_x^3\alpha_{xt}|w|^2dxdt\right| \le \frac{4\cdot 8^4}{3}\frac{T}{\varepsilon L^2 s^3}\varepsilon^2\iint_Q s^7L^6\beta^7|w|^2dxdt.$$

- $\left| -\frac{s}{2} \iint\limits_Q \alpha_{tt} |w|^2 dxdt \right| \le \frac{22}{9} \frac{T^2}{\varepsilon^2 L^4 s^6} \varepsilon^2 \iint\limits_Q s^7 L^6 \beta^7 |w|^2 dxdt.$

- $\left| -12\varepsilon s^2 \iint\limits_Q \alpha_x \alpha_{xt} |w_x|^2 dxdt \right| \le 4 \cdot 8^2 \frac{T}{\varepsilon L^2 s^3} \varepsilon^2 \iint\limits_Q s^5 L^4 \beta^5 |w_x|^2 dxdt.$

- $\left| -sM \iint\limits_Q \alpha_{xt} |w|^2 dxdt \right| \le \frac{2}{3} \frac{T}{\varepsilon L^2 s^3} \frac{|M| T^2}{\varepsilon L^3 s^3} \varepsilon^2 \iint\limits_Q s^7 L^6 \beta^7 |w|^2 dxdt.$

- $\left| 24\varepsilon^2 s^3 \iint\limits_Q \alpha_{xx}^3 |w_x|^2 dxdt \right| \le \frac{12}{4^{2/3}} \frac{T^{4/3}}{L^4 s^2} \varepsilon^2 \iint\limits_Q s^5 L^4 \beta^5 |w_x|^2 dxdt.$

- $\left| 4\varepsilon s^4 M \iint\limits_Q \alpha_x^3 \alpha_{xx} |w|^2 dxdt \right| \le 8^3 \frac{|M| T^2}{\varepsilon L^3 s^3} \varepsilon^2 \iint\limits_Q s^7 L^6 \beta^7 |w|^2 dxdt.$

- $-12\varepsilon^2 s^5 \iint\limits_Q \alpha_x^2 \alpha_{xx}^3 |w|^2 dxdt \ge 0.$

- $\left| 6\varepsilon s^2 \iint\limits_Q \alpha_{xxt} \alpha_{xx} |w|^2 dxdt \right| \le \frac{2}{4^{2/3}} \frac{T}{\varepsilon L^2 s^3} \frac{T^{4/3}}{L^4 s^2} \varepsilon^2 \iint\limits_Q s^7 L^6 \beta^7 |w|^2 dxdt.$

- $\left| -6\varepsilon s^2 \iint\limits_Q \alpha_t \alpha_{xx} |w_x|^2 dxdt \right| \le 16 \frac{T}{\varepsilon L^2 s^3} \varepsilon^2 \iint\limits_Q s^5 L^4 \beta^5 |w_x|^2 dxdt.$

- $\left| -18\varepsilon s^2 M \iint\limits_Q \alpha_x \alpha_{xx} |w_x|^2 dxdt \right| \le 36 \frac{|M| T^2}{\varepsilon L^3 s^3} \varepsilon^2 \iint\limits_Q s^5 L^4 \beta^5 |w_x|^2 dxdt.$

- $-\frac{sM^2}{2} \iint\limits_Q \alpha_{xx} |w|^2 dxdt \ge 0.$

In (A.3) we employ (A.8) and the previous points to obtain

$$\mathcal{D}(w) \geq \mathcal{D}_0(s)\varepsilon^2 \iint_Q s^7 L^6 \beta^7 |w|^2 dxdt + \mathcal{D}_1(s)\varepsilon^2 \iint_Q s^5 L^4 \beta^5 |w_x|^2 dxdt$$
$$+ C\varepsilon^2 \iint_Q s^3 L^2 \beta^3 |w_{xx}|^2 dxdt + C\varepsilon^2 \iint_Q s\beta |w_{xxx}|^2 dxdt, \tag{A.9}$$

where

$$\mathcal{D}_0(s) = 8\cdot 7^6 - \frac{4\cdot 8^4}{3}\frac{T}{\varepsilon L^2 s^3} - \frac{22}{9}\frac{T^2}{\varepsilon^2 L^4 s^6}$$
$$- \frac{2}{3}\frac{T}{\varepsilon L^2 s^3}\frac{|M|T^2}{\varepsilon L^3 s^3} - 8^3\frac{|M|T^2}{\varepsilon L^3 s^3} - \frac{2}{4^{2/3}}\frac{T}{\varepsilon L^2 s^3}\frac{T^{4/3}}{L^4 s^2}, \tag{A.10}$$
$$\mathcal{D}_1(s) = 18\cdot 7^4 - 4\cdot 8^2\frac{T}{\varepsilon L^2 s^3} - \frac{12}{4^{2/3}}\frac{T^{4/3}}{L^4 s^2} - 16\frac{T}{\varepsilon L^2 s^3} - 36\frac{|M|T^2}{\varepsilon L^3 s^3}. \tag{A.11}$$

In order to handle (A.10) and (A.11), we consider $s > 0$ such that

$$s \geq C_1 T^{1/3} L^{-2/3}\varepsilon^{-1/3} + C_2 T^{2/3} L^{-1}\varepsilon^{-1/3}|M|^{1/3} + C_3 T^{2/3} L^{-2}, \tag{A.12}$$

where $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$. In view of (4.11) of the proof of Theorem 1.2, we would like to choose $C_1 > 0$ and $C_2 > 0$ as small as possible, that is to say, the constants multiplying $\varepsilon^{-1/3}$. Note that (A.12) implies

$$\frac{T}{\varepsilon L^2 s^3} \leq \frac{1}{C_1^3}, \quad \frac{|M|T^2}{\varepsilon L^3 s^3} \leq \frac{1}{C_2^3}, \quad \frac{T^{4/3}}{L^4 s^2} \leq \frac{1}{C_3^2}, \tag{A.13}$$

and hence, from (A.10) and (A.11) we get

$$\mathcal{D}_0(s) \geq 8\cdot 7^6 - \frac{4}{3}\cdot 8^4\frac{1}{C_1^3} - \frac{22}{9}\frac{1}{C_1^6} - \frac{2}{3}\frac{1}{C_1^3}\frac{1}{C_2^3} - 8^3\frac{1}{C_2^3} - \frac{2}{4^{2/3}}\frac{1}{C_1^3}\frac{1}{C_3^2}, \tag{A.14}$$
$$\mathcal{D}_1(s) \geq 18\cdot 7^4 - 4\cdot 8^2\frac{1}{C_1^3} - \frac{12}{4^{2/3}}\frac{1}{C_3^2} - 14\frac{1}{C_1^3} - 36\frac{1}{C_2^3}. \tag{A.15}$$

After the combination of (A.9) together with (A.14) and (A.15), it can be checked that the choice of $C_1 = 9/40$, $C_2 = 1/8$ and $C_3 > 0$ large enough give us

$$\mathcal{D}(w) \geq C\varepsilon^2 \iint_Q \left(L^6 s^7 \beta^7 |w|^2 + L^4 s^5 \beta^5 |w_x|^2 + L^2 s^3 \beta^3 |w_{xx}|^2 + s\beta^2 |w_{xxx}|^2\right) dxdt. \tag{A.16}$$

**Handling $\mathcal{B}(w, x)$ for $x \in \{0, L\}$.** Let us focus on the case $x = L$. We first handle the first three terms in (A.4), which are the leading terms among the boundary terms. Plugging $\alpha_x(t, L) = 7L\beta(t)$ into them yields

$$10\varepsilon^2 s^5 \int_0^T \alpha_x(t,L)^5 |w_x(t,L)|^2 dt + 10\varepsilon^2 s^3 \int_0^T \alpha_x(t,L)^3 |w_{xx}(t,L)|^2 dt$$

$$+ 2\varepsilon^2 s \int_0^T \alpha_x(t,L) |w_{xxx}(t,L)|^2 dt = 10 \cdot 7^5 \varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dt$$

$$+ 10 \cdot 7^3 \varepsilon^2 \int_0^T s^3 L^3 \beta(t)^3 |w_{xx}(t,L)|^2 dt + 14\varepsilon^2 \int_0^T sL\beta(t) |w_{xxx}(t,L)|^2 dt. \quad \text{(A.17)}$$

These terms, as in the previous part of the proof, will allow us to absorb the remaining terms in (A.4). We pay special attention to the fourth and sixth terms in (A.4), where it will be needed

$$w_{xx}(t,x) = -2s\alpha_x(t,x) w_x(t,x), \quad (t,x) \in [0,T] \times \{0,L\},$$

which is a consequence of $w = e^{-s\alpha} q$ and the boundary conditions of adjoint equation (1.4). The following inequalities can be obtained by using the weight function defined in (3.1) together with (A.6) and (A.7).

- $$\left| \varepsilon \int_0^T w_{xt}(t,L) w_{xx}(t,L)\, dt \right| \le \frac{7}{4^{1/3} \cdot 3} \frac{T}{\varepsilon L^2 s^3} \frac{T^{2/3}}{L^2 s} \varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dt.$$

- $$\left| -24\varepsilon^2 s^3 \int_0^T \alpha_x(t,L) \alpha_{xx}(t,L)^2 |w_x(t,L)|^2 dt \right|$$
$$\le \frac{168}{4^{2/3}} \frac{T^{4/3}}{L^4 s^2} \varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dxdt.$$

- $$30\varepsilon^2 s^3 \int_0^T \alpha_x(t,L)^2 \alpha_{xx}(t,L) w_x(t,L) w_{xx}(t,L)\, dt \ge 0.$$

- $$\left| 4\varepsilon^2 s^3 \int_0^T \alpha_x(t,L)^3 w_x(t,L) w_{xxx}(t,L)\, dt \right| \le 4 \cdot 7^5 \varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dt$$
$$+ 7\varepsilon^2 \int_0^T sL\beta(t) |w_{xxx}(t,L)|^2 dt.$$

- $$\left|2\varepsilon s^2 \int_0^T \alpha_t(t,L)\alpha_x(t,L)|w_x(t,L)|^2 dt\right| \le \frac{112}{3}\frac{T}{\varepsilon L^2 s^3}\varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dxdt.$$

- $$\left|-\varepsilon s^2 M \int_0^T \alpha_x(t,L)^2 |w_x(t,L)|^2 dt\right| \le \frac{49}{4}\frac{|M|T^2}{\varepsilon L^3 s^3}\varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dxdt.$$

- $$\begin{aligned}\left|-\varepsilon M \int_0^T w_x(t,L) w_{xxx}(t,L)\, dt\right| \le \frac{1}{4^3\cdot 7}\frac{|M|^2 T^4}{\varepsilon^2 L^6 s^6}\varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dt \\ + 7\varepsilon^2 \int_0^T sL\beta(t)|w_{xxx}(t,L)|^2 dt.\end{aligned}$$

- $$\left|\frac{\varepsilon M}{2} \int_0^T |w_{xx}(t,L)|^2 dt\right| \le \frac{1}{8}\frac{|M|T^2}{\varepsilon L^3 s^3}\varepsilon^2 \int_0^T s^3 L^3 \beta(t)^3 |w_{xx}(t,L)|^2 dxdt.$$

In (A.4) we employ (A.17) and the previous points to obtain

$$\mathcal{B}(w,L) \ge \mathcal{B}_1(s)\varepsilon^2 \int_0^T s^5 L^5 \beta(t)^5 |w_x(t,L)|^2 dt + \mathcal{B}_2(s)\varepsilon^2 \int_0^T s^3 L^3 \beta(t)^3 |w_{xx}(t,L)|^2 dt, \quad \text{(A.18)}$$

where

$$\begin{aligned}\mathcal{B}_1(s) = 6\cdot 7^5 - \frac{7}{4^{1/3}\cdot 3}\frac{T}{\varepsilon L^2 s^3}\frac{T^{2/3}}{L^2 s} - \frac{168}{4^{2/3}}\frac{T^{4/3}}{L^4 s^2} - \frac{112}{3}\frac{T}{\varepsilon L^2 s^3} \\ - \frac{49}{4}\frac{|M|T^2}{\varepsilon L^3 s^3} - \frac{1}{4^3\cdot 7}\frac{|M|^2 T^4}{\varepsilon^2 L^6 s^6},\end{aligned} \quad \text{(A.19)}$$

$$\mathcal{B}_2(s) = 10\cdot 7^3 - \frac{1}{8}\frac{|M|T^2}{\varepsilon L^3 s^3}. \quad \text{(A.20)}$$

In order to handle (A.19) and (A.20), we consider $s > 0$ as in (A.12). As we did in the previous part of the proof, in view of (4.11) of the proof of Theorem 1.2, we would like to choose $C_1 > 0$ and $C_2 > 0$ as small as possible, that is to say, the constants multiplying $\varepsilon^{-1/3}$. Note that (A.13) implies

$$\mathcal{B}_1(s) \ge 6\cdot 7^5 - \frac{7}{4^{1/3}\cdot 3}\frac{1}{C_1^3}\frac{1}{C_3^2} - \frac{168}{4^{2/3}}\frac{1}{C_3^2} - \frac{112}{3}\frac{T}{\varepsilon L^2 s^3} - \frac{49}{4}\frac{1}{C_2^3} - \frac{1}{4^3\cdot 7}\frac{1}{C_2^6}, \quad \text{(A.21)}$$

$$\mathcal{B}_1(2) \ge 10\cdot 7^3 - \frac{1}{8}\frac{1}{C_2^3}. \quad \text{(A.22)}$$

After the combination of (A.18) together with (A.21) and (A.22), it can be checked that the choice of $C_1 = 9/40$, $C_2 = 1/8$ and $C_3 > 0$ large enough give us

$$\mathcal{B}(w, L) \geq 0. \tag{A.23}$$

Let us focus on the case $x = 0$. Similar arguments as those used to handle the case $x = L$ allow us to conclude that if (A.12) holds with $C_1 = 9/40$, $C_2 = 1/8$ and $C_3 > 0$ large enough, then we get

$$|\mathcal{B}(w, 0)| \leq C\varepsilon^2 \int_0^T \left(L^5 s^5 \beta(t)^5 |w_x(t, 0)|^2 + Ls\beta(t)|w_{xxx}(t, 0)|^2\right) dt. \tag{A.24}$$

**Gathering terms.** From now on and until the end of the proof, we consider

$$s \geq \frac{9}{40} T^{1/3} L^{-2/3} \varepsilon^{-1/3} + \frac{1}{8} T^{2/3} L^{-1} \varepsilon^{-1/3} |M|^{1/3} + C T^{2/3} L^{-2},$$

with the constant $C > 0$ being as large as needed. On the one hand, from the combination of (A.5), (A.16), (A.23) and (A.24), we arrive at

$$\frac{1}{C} \iint_Q \left(L^6 s^7 \beta^7 |w|^2 + L^4 s^5 \beta^5 |w_x|^2 + L^2 s^3 \beta^3 |w_{xx}|^2 + s\beta |w_{xxx}|^2\right) dx dt$$
$$\leq \frac{1}{\varepsilon^2} (\mathcal{P}_1 w, \mathcal{P}_2 w)_{L^2(Q)} + C \int_0^T \left(L^5 s^5 \beta(t)^5 |w_x(t, 0)|^2 + Ls\beta(t)|w_{xxx}(t, 0)|^2\right) dt. \tag{A.25}$$

On the other hand, by plugging $\alpha_{xx}(t, x) = -\beta(t)$ into (A.1), then employing the Cauchy inequality we obtain

$$\frac{1}{\varepsilon^2} \|\mathcal{P}_3 w\|^2_{L^2(Q)} \leq C \frac{T^2}{L^6 s^3} \iint_Q L^6 s^7 \beta^7 |w|^2 dx dt + C \frac{T^{2/3}}{L^2 s} \iint_Q L^2 s^3 \beta^3 |w_{xx}|^2 dx dt.$$

Therefore, by taking into account (A.2), (A.25) and the previous inequality, we deduce

$$\iint_Q \left(L^6 s^7 \beta^7 |w|^2 + L^4 s^5 \beta^5 |w_x|^2 + L^2 s^3 \beta^3 |w_{xx}|^2 + s\beta |w_{xxx}|^2\right) dx dt$$
$$\leq \frac{C}{\varepsilon^2} \|\mathcal{P} w\|^2_{L^2(Q)} + C \int_0^T \left(L^5 s^5 \beta(t)^5 |w_x(t, 0)|^2 + Ls\beta(t)|w_{xxx}(t, 0)|^2\right) dt. \tag{A.26}$$

**Conclusion**. As we did in the previous parts, we can get the following inequalities.

- $$\iint\limits_{Q} e^{-2s\alpha}\left(L^6 s^7\beta^7\left|e^{s\alpha}w\right|^2 + L^4 s^5\beta^5\left|\left(e^{s\alpha}w\right)_x\right|^2 + L^2 s^3\beta^3\left|\left(e^{s\alpha}w\right)_{xx}\right|^2\right)dxdt$$
$$\le C\frac{T^{4/3}}{L^4 s^2}\iint\limits_{Q} L^6 s^7\beta^7|w|^2 dxdt + C\sum_{k=0}^{2}\iint\limits_{Q} L^{6-2k}s^{7-2k}\beta^{7-2k}\left|\frac{\partial^k w}{\partial x^k}\right|^2 dxdt.$$
- $$\iint\limits_{Q} e^{-2s\alpha}s\alpha\left|\left(e^{s\alpha}w\right)_{xxx}\right|^2 dxdt \le C\frac{T^{4/3}}{L^4 s^2}\iint\limits_{Q} L^6 s^7\beta^7|w|^2 dxdt$$
$$+ C\frac{T^{4/3}}{L^4 s^2}\iint\limits_{Q} L^4 s^5\beta^5|w_x|^2 dxdt + C\sum_{k=0}^{3}\iint\limits_{Q} L^{6-2k}s^{7-2k}\beta^{7-2k}\left|\frac{\partial^k w}{\partial x^k}\right|^2 dxdt.$$

Accordingly, the preceding inequalities tell us that

$$\sum_{k=0}^{3}\iint\limits_{Q} e^{-2s\alpha}L^{6-2k}s^{7-2k}\beta^{7-2k}\left|\frac{\partial^k(e^{s\alpha}w)}{\partial x^k}\right|^2 dxdt \le C\sum_{k=0}^{3}\iint\limits_{Q} L^{6-2k}s^{7-2k}\beta^{7-2k}\left|\frac{\partial^k w}{\partial x^k}\right|^2 dxdt.$$

Finally, combining this inequality with (A.26), and then considering $w = e^{-s\alpha}q$ and $\mathcal{P}w := e^{-s\alpha}\mathcal{L}(e^{s\alpha}w)$, we arrive at the desired Carleman estimate. The proof of Proposition 3.1 is complete. □

## References


[1] L. Baudouin, E. Cerpa, E. Crépeau, A. Mercado, Lipschitz stability in an inverse problem for the Kuramoto–Sivashinsky equation, Appl. Anal. 92 (10) (2013) 2084–2102.

[2] N. Carreño, E. Cerpa, Local controllability of the stabilized Kuramoto–Sivashinsky system by a single control acting on the heat equation, J. Math. Pures Appl. 106 (4) (2016) 670–694.

[3] N. Carreño, S. Guerrero, On the non-uniform null controllability of a linear KdV equation, Asymptot. Anal. 94 (1–2) (2015) 33–69.

[4] E. Cerpa, Null controllability and stabilization of the linear Kuramoto–Sivashinsky equation, Commun. Pure Appl. Anal. 9 (1) (2010) 91–102.

[5] E. Cerpa, A. Mercado, Local exact controllability to the trajectories of the 1-D Kuramoto–Sivashinsky equation, J. Differential Equations 250 (4) (2011) 2024–2044.

[6] E. Cerpa, P. Guzmán, A. Mercado, On the control of the linear Kuramoto–Sivashinsky equation, ESAIM Control Optim. Calc. Var. (2015), http://dx.doi.org/10.1051/cocv/2015044, in press.

[7] J.-M. Coron, Control and Nonlinearity, Mathematical Surveys and Monographs, vol. 136, American Mathematical Society, Providence, RI, 2007.

[8] J.-M. Coron, S. Guerrero, Singular optimal control: a linear 1-D parabolic–hyperbolic example, Asymptot. Anal. 44 (3–4) (2005) 237–257.

[9] R. Dautray, J.-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, vol. 3, Springer Verlag, Berlin, Heidelberg, 2000.

[10] L. Evans, Partial Differential Equations, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 1998.

[11] H.O. Fattorini, D.L. Russell, Exact controllability theorems for linear parabolic equation in one space dimension, Arch. Ration. Mech. Anal. 43 (4) (1971) 272–292.

[12] E. Fernández-Cara, S. Guerrero, O.Y. Imanuvilov, J.-P. Puel, Local exact controllability of the Navier–Stokes system, J. Math. Pures Appl. 83 (12) (2004) 1501–1542.
[13] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series, vol. 34, Seoul National University, Republic of Korea, 1996.
[14] P. Gao, Insensitizing controls for the Cahn–Hilliard type equation, Electron. J. Qual. Theory Differ. Equ. 35 (2014) 1–22.
[15] P. Gao, A new global Carleman estimate for the one-dimensional Kuramoto–Sivashinsky equation and applications to exact controllability to the trajectories and an inverse problem, Nonlinear Anal. 117 (2015) 133–147.
[16] P. Gao, A new global Carleman estimate for Cahn–Hilliard type equation and its applications, J. Differential Equations 260 (1) (2016) 427–444.
[17] O. Glass, A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit, J. Funct. Anal. 258 (3) (2010) 852–868.
[18] O. Glass, S. Guerrero, Some exact controllability results for the linear KdV equation and uniform controlability in the zero-dispersion limit, Asymptot. Anal. 60 (1–2) (2008) 61–100.
[19] O. Glass, S. Guerrero, Uniform controllability of a transport equation in zero diffusion-dispersion limit, Math. Models Methods Appl. Sci. 19 (9) (2009) 1567–1601.
[20] P. Guzmán, Lipschitz stability in an inverse problem for the main coefficient of a Kuramoto–Sivashinsky type equation, J. Math. Anal. Appl. 408 (1) (2013) 275–290.
[21] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems, SIAM Rev. 30 (1) (1988) 1–68.
[22] P. Lissy, A link between the cost of fast controls for the 1-D heat equation and the uniform controllability of a 1-D transport-diffusion equation, C. R. Math. Acad. Sci. Paris 350 (11–12) (2012) 591–595.
[23] P. Lissy, Explicit lower bounds for the cost of fast controls for some 1-D parabolic or dispersive equations, and a new lower bound concerning the uniform controllability of the 1-D transport-diffusion equation, J. Differential Equations 259 (10) (2015) 5331–5352.
[24] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathematical Sciences, vol. 68, Springer Verlag, New York, 1997.
[25] M. Tucsnak, G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser Verlag, Basel, Boston, Berlin, 2009.
[26] Z. Zhou, Observability estimate and null controllability for one-dimensional fourth order parabolic equation, Taiwanese J. Math. 16 (6) (2012) 1991–2017.