J. Math. Anal. Appl. 425 (2015) 655–665

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Exact boundary controllability of a microbeam model

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A R T I C L E   I N F O

Article history:
Received 2 September 2014
Available online 7 January 2015
Submitted by D.L. Russell

Keywords:
Microbeam model
Hyperbolic equation
Boundary control
Exact controllability
Multiplier method

A B S T R A C T

In this paper we study the exact controllability property of a microbeam model by means of a single boundary control. We use the multiplier method together with the controllability–observability duality to obtain a time $T^* > 0$ such that the corresponding linear equation is exactly controllable provided that the control time is greater than $T^* > 0$. Our exact controllability result improves the one obtained in Vatankhah et al. (2014) [20], which uses six boundary controls instead of a single one.

1. Introduction

A microbeam is a thin beam with cross-sectional area in the order of a few square microns and length in the order of approximately ten to hundreds of microns. The microbeams are a major structural component of micro-electromechanical (MEMS) devices such as electrostatic actuators [3], micro-switches [9] and mechanical resonators [17] for instance.

In this paper we study the exact controllability property of a microbeam model by means of a single boundary control. Denoting by $z = z(t, x)$ the later deflection of a microbeam, the model that we consider here has been obtained in [12, Section 4.1] and [21, Section 2] by using the modified strain gradient theory developed in [13] together with Hamilton's Principle, namely:

$$\begin{cases} \rho A z_{tt} + S z_{xxxx} - K z_{xxxxxx} = 0, & (t,x) \in (0,T) \times (0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ z_x(t,0) = 0, \quad z_x(t,L) = 0, & t \in (0,T), \\ z_{xx}(t,0) = 0, \quad z_{xx}(t,L) = u(t), & t \in (0,T), \\ z(0,x) = z_0(x), \quad z_t(0,x) = z_1(x), & x \in (0,L). \end{cases} \tag{1}$$

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http://dx.doi.org/10.1016/j.jmaa.2014.12.059
0022-247X/

The parameters $\rho > 0$, $A > 0$ and $L > 0$ are, respectively, the density, cross-sectional area and the length of the microbeam. The physical characterization of the microbeam is contained in the parameters

$$S = EI + \mu A\left(2l_0^2 + \frac{43}{225}l_1^2 + l_2^2\right), \quad K = \mu A\left(2l_0^2 + \frac{4}{5}l_1^2\right),$$

where $l_0 > 0$, $l_1 > 0$ and $l_2 > 0$ correspond to the independent material parameters associated with dilatation gradients, deviatoric stretch gradients and symmetric rotation gradients respectively. In fact, these parameters, introduced in [13, Section 2.4], characterize the size effects phenomenon of the beam when its structural size is in the order of microns. As usual, $E > 0$ is the Young modulus, $I > 0$ the area moment of inertia of the microbeam cross-section and $\mu > 0$ the shear modulus.

Remark 1.1. When the structural size of the beam is no longer in the order of microns, we may consider $l_0 = l_1 = l_2 = 0$, obtaining the Euler–Bernouilli beam equation.

Given the hyperbolic character of Eq. (1), the appropriate control notion to study is the exact controllability, which is defined as follows. Eq. (1) is said to be exactly controllable in time $T > 0$ if given any initial state (z_0, z_1) and any final state $(\widetilde{z}_0, \widetilde{z}_1)$ there exists a control u such that its corresponding solution $z = z(t, x)$ satisfies $z(T, \cdot) = \widetilde{z}_0(\cdot)$ and $z_t(T, \cdot) = \widetilde{z}_1(\cdot)$.

We use the multiplier method together with the controllability–observability duality (e.g. [6, Theorem 2.44] and [19, Theorem 11.2.1]) to obtain a time $T^* > 0$ such that Eq. (1) is exactly controllable in time $T > T^*$. This is our main result.

Theorem 1.1. *Let us assume that*

$$T > T^* := 2L \max\left\{1, \frac{L^2}{\pi^2}\frac{\rho A}{S}\right\}.$$

Then, for every $(z_0, z_1) \in L^2(0, L) \times H^{-3}(0, L)$ and every $(\widetilde{z}_0, \widetilde{z}_1) \in L^2(0, L) \times H^{-3}(0, L)$ there exists $u \in L^2(0, T)$ such that the unique solution $z \in C([0, T]; L^2(0, L)) \cap C^1([0, T]; H^{-3}(0, L))$ defined by transposition of Eq. (1) satisfies $z(T, \cdot) = \widetilde{z}_0(\cdot)$ in $L^2(0, L)$ and $z_t(T, \cdot) = \widetilde{z}_1(\cdot)$ in $H^{-3}(0, L)$.

This result improves [20, Theorem 3], where it is addressed the same control problem but with six boundary controls. Nevertheless, we conjecture that our result might be improved, by following the strategies in [7], in the sense that Theorem 1.1 would actually hold for $T^* = 0$. We will present this with further details as an open problem in Section 4.

This paper is organized as follows. In Section 2 we present the well-posedness results needed for studying control system (1). In Section 3 we prove the exact controllability property for control system (1) given by Theorem 1.1. Finally, in Section 4 we suggest the above-mentioned open problem.

2. Well-posedness

The purpose of this section is to present the well-posedness results needed for studying control system (1). In virtue of the control framework that we shall adopt in Section 3, it is necessary to have solutions for Eq. (1) with data $u \in L^2(0, T)$ and $(z_0, z_1) \in L^2(0, L) \times H^{-3}(0, L)$. Therefore, the suitable notion of solutions for this equation, given the previous set of data, are the solutions defined by transposition. We shall proceed this study as in [10, Chapter 2] for instance, and hence, we first must know about the solutions of this equation when the data is regular enough.

2.1. Finite energy solutions

Let us consider the equation

$$\begin{cases} \rho A z_{tt} + S z_{xxxx} - K z_{xxxxxx} = f, & (t,x) \in (0,T)\times(0,L), \\ z(t,0) = 0, \quad z(t,L) = 0, & t \in (0,T), \\ z_x(t,0) = 0, \quad z_x(t,L) = 0, & t \in (0,T), \\ z_{xx}(t,0) = 0, \quad z_{xx}(t,L) = u(t), & t \in (0,T), \\ z(0,x) = z_0(x), \quad z_t(0,x) = z_1(x), & x \in (0,L). \end{cases} \tag{2}$$

For a regular enough solution of Eq. (2) we define its energy as

$$E(t) := \frac{1}{2}\int_0^L \left(\left|z_t(t,x)\right|^2 + \frac{S}{\rho A}\left|z_{xx}(t,x)\right|^2 + \frac{K}{\rho A}\left|z_{xxx}(t,x)\right|^2 \right) dx, \quad t \in [0,T]. \tag{3}$$

We first study Eq. (2) with homogeneous boundary conditions. By using the variational approach for evolution equations, which has been developed in [15, Chapter 3, Section 8] for instance, we have obtained the following result.

Proposition 2.1. *Let $f \in L^2(0,T;L^2(0,L))$ and $(z_0,z_1) \in H_0^3(0,L)\times L^2(0,L)$. Then, Eq. (2) with $u=0$ has a unique solution $z \in C([0,T];H_0^3(0,L)) \cap C^1([0,T];L^2(0,L))$. Moreover, there exists $C = C(T,L,\rho,A,S,K) > 0$ such that*

$$\|z\|_{C([0,T];H_0^3(0,L))\cap C^1([0,T];L^2(0,L))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \left\|(z_0,z_1)\right\|_{H_0^3(0,L)\times L^2(0,L)}\right). \tag{4}$$

Proof. Let us consider the bilinear form $a : H_0^3(0,L)\times H_0^3(0,L) \to \mathbb{R}$ defined by

$$a(u,v) := \frac{S}{\rho A}\int_0^L u''(x)v''(x)\,dx + \frac{K}{\rho A}\int_0^L u'''(x)v'''(x)\,dx,$$

which turns out to be continuous thanks to the Cauchy–Schwarz inequality. Furthermore, the Poincaré inequality tells us that there exists $C = C(L,\rho,A,S,K) > 0$ such that

$$a(v,v) = \frac{S}{\rho A}\int_0^L \left|v''(x)\right|^2 dx + \frac{K}{\rho A}\int_0^L \left|v'''(x)\right|^2 dx \ge C\|v\|^2_{H_0^3(0,L)}.$$

Accordingly, [15, Theorem 8.2, Chapter 3] leads us to the existence of a unique solution $z \in C([0,T]; H_0^3(0,L)) \cap C^1([0,T];L^2(0,L))$ to Eq. (2) when $u = 0$.

Now we proceed to obtain (4). Some integrations by parts on $(0,L)$ allow us to prove that for every $t \in [0,T]$ the solution $z = z(t,x)$ of Eq. (2) with $u = 0$ satisfies

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_0^L \left(z_t(t,x)z_{tt}(t,x) + \frac{S}{\rho A} z_{xx}(t,x)z_{xxt}(t,x) + \frac{K}{\rho A} z_{xxx}(t,x)z_{xxxt}(t,x) \right) dx, \\ &= \int_0^L \left(z_{tt}(t,x) + \frac{S}{\rho A} z_{xxxx}(t,x) - \frac{K}{\rho A} z_{xxxxxx}(t,x) \right) z_t(t,x)\,dx, \end{aligned}$$

$$\le \frac{1}{2\rho A}\left\|f(t,\cdot)\right\|^2_{L^2(0,L)} + \frac{1}{2\rho A}E(t).$$

Here we apply the Grönwall lemma together with the Poincaré inequality to get (4). The proof of Proposition 2.1 is complete. □

Remark 2.1. Let $f = 0$, $u = 0$ and $(z_0, z_1) \in H^3_0(0,L) \times L^2(0,L)$. It is straightforward to see from the proof of the previous proposition that the conservation of energy holds. That is to say, for every $t \in [0,T]$ it holds

$$E(t) = E(0) := \frac{1}{2}\int_0^L \left(\left|z_1(x)\right|^2 + \frac{S}{\rho A}\left|z_0''(x)\right|^2 + \frac{K}{\rho A}\left|z_0'''(x)\right|^2\right) dx. \tag{5}$$

Moreover, thanks to this conservation of energy and the Poincaré inequality, there exists $C > 0$ such that for every $t \in [0,T]$ it holds

$$\left\|(z_0, z_1)\right\|_{H^3_0(0,L)\times L^2(0,L)} \le C\left\|\left(z(t,\cdot), z_t(t,\cdot)\right)\right\|_{H^3_0(0,L)\times L^2(0,L)}. \tag{6}$$

In order to obtain more precise information about the solutions of Eq. (1) with regular data, we derive the following key identity, that later we will also use in studying the control properties of control system (1).

Lemma 2.1. *Let $(z_0, z_1) \in H^3_0(0,L) \times L^2(0,L)$. Then, the unique solution $z \in C([0,T]; H^3_0(0,L)) \cap C^1([0,T]; L^2(0,L))$ of Eq. (2) with $f = 0$ and $u = 0$ satisfies the identity*

$$\begin{aligned} &\frac{L}{2}\frac{K}{\rho A}\int_0^T \left|z_{xxx}(t,L)\right|^2 dt - \int_0^L x z_x(t,x) z_t(t,x)\Big|_0^T dx \\ &= \frac{1}{2}\int_0^T\int_0^L \left(\left|z_t(t,x)\right|^2 + \frac{3S}{\rho A}\left|z_{xx}(t,x)\right|^2 + \frac{5K}{\rho A}\left|z_{xxx}(t,x)\right|^2\right) dx dt. \end{aligned} \tag{7}$$

Proof. The unique solution $z \in C([0,T]; H^3_0(0,L)) \cap C^1([0,T]; L^2(0,L))$ for Eq. (2) is given by Proposition 2.1. The proof of this lemma is done by using the multiplier method. Multiplying Eq. (2) with $f = 0$ and $u = 0$ by $x z_x(t,x)$ we get

$$\int_0^T\int_0^L \left(z_{tt}(t,x) + \frac{S}{\rho A} z_{xxxx}(t,x) - \frac{K}{\rho A} z_{xxxxxx}(t,x)\right) x z_x(t,x)\, dx dt = 0. \tag{8}$$

Some integrations by parts on $(0,T)$ or $(0,L)$ give us the following expressions.

- $\int_0^T \int_0^L z_{tt}(t,x) x z_x(t,x)\, dx dt = \frac{1}{2}\int_0^T \int_0^L |z_t(t,x)|^2 dx dt + \int_0^L x z_t(t,x) z_x(t,x)|_0^T\, dx.$
- $\int_0^T \int_0^L \frac{S}{\rho A} z_{xxxx}(t,x) x z_x(t,x)\, dx dt = \frac{1}{2}\int_0^T \int_0^L \frac{3S}{\rho A}|z_{xx}(t,x)|^2 dx dt.$
- $-\int_0^T \int_0^L \frac{K}{\rho A} z_{xxxxxx}(t,x) x z_x(t,x)\, dx dt = \frac{1}{2}\int_0^T \int_0^L \frac{5K}{\rho A}|z_{xxx}(t,x)|^2 dx dt - \frac{L}{2}\frac{K}{\rho A}\int_0^T |z_{xxx}(t,L)|^2 dt.$

Therefore, from the combination of the three previous expressions with (8) we arrive at identity (7). The proof of Lemma 2.1 is complete. □

The above-mentioned precise information about the solutions of Eq. (1) with regular data is the following one.

Lemma 2.2. *Let* $(z_0, z_1) \in H_0^3(0,L) \times L^2(0,L)$. *Then, the unique solution* $z \in C([0,T]; H_0^3(0,L)) \cap C^1([0,T]; L^2(0,L))$ *of Eq.* (2) *with* $f = 0$ *and* $u = 0$ *satisfies* $z_{xxx}(\cdot, L) \in L^2(0,T)$. *Moreover, there exists* $C = C(T, L, \rho, A, S, K) > 0$ *such that*

$$\int_0^T \left|z_{xxx}(t,L)\right|^2 dt \le C\left\|(z_0, z_1)\right\|^2_{H_0^3(0,L)\times L^2(0,L)}. \tag{9}$$

Proof. The proof of this lemma is based on identity (7). In fact, from that identity, the Cauchy–Schwarz inequality and the Poincaré inequality it follows that

$$\begin{aligned}
\frac{L}{2}\frac{K}{\rho A}\int_0^T \left|z_{xxx}(t,L)\right|^2 dt &\le \int_0^L x z_x(t,x) z_t(t,x)\Big|_0^T dx + 5\int_0^T E(t)\,dt, \\
&\le L\left\|\int_0^L \left(\left|z_t(t,x)\right|^2 + \left|z_x(t,x)\right|^2\right) dx\right\|_{L^\infty(0,T)} + 5\int_0^T E(t)\,dt, \\
&\le C\left\|E(t)\right\|_{L^\infty(0,T)} + 5\int_0^T E(t)\,dt.
\end{aligned}$$

Recall that $E(t)$ was defined in (3). Therefore, the above inequality together with conservation of energy (5) gives us the desired result. The proof of Lemma 2.2 is complete. □

Remark 2.2. The regularity for $z_{xxx}(\cdot, L)$ obtained in the previous lemma cannot be deduced from trace theorems (e.g. [15, Chapter 1]). For this reason, this extra regularity is known as hidden regularity. This kind of result is usual when dealing with hyperbolic equations (e.g. [10] and [14]), or equations with similar properties such as the Kawahara (e.g. [2] and [8]), Korteweg–de Vries (e.g. [5] and [18]) and the Schrödinger (e.g. [1] and [16]) equations.

We can use a suitable lifting function together with Proposition 2.1 to study Eq. (2) with non-homogeneous boundary conditions. To this end, let us introduce the space

$$H_l^3(0,L) := \left\{y \in H^3(0,L) \cap H_0^2(0,L) / y''(0) = 0\right\}, \tag{10}$$

which is well-defined thanks to the continuous injection $H^3(0,L) \hookrightarrow C^2([0,L])$.

Proposition 2.2. *Let* $f \in L^2(0,T; L^2(0,L))$, $u \in \{u \in C^2([0,T]) / u(0) = u'(0) = 0\}$ *and* $(z_0, z_1) \in H_0^3(0,L) \times L^2(0,L)$. *Then, Eq.* (2) *has a unique solution* $z \in C([0,T]; H_l^3(0,L)) \cap C^1([0,T]; L^2(0,L))$. *Moreover, there exists* $C = C(T, L, \rho, A, S, K) > 0$ *such that*

$$\|z\|_{C([0,T];H_l^3(0,L))\cap C^1([0,T];L^2(0,L))} \le C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|u\|_{C^2([0,T])} + \left\|(z_0, z_1)\right\|_{H_0^3(0,L)\times L^2(0,L)}\right). \tag{11}$$

Proof. We define the lifting function

$$\psi(t,x) := \frac{x^3(L-x)^2}{2L^3} u(t).$$

By taking into account that $g := f - \rho A\psi_{tt} - S\psi_{xxxx} + K\psi_{xxxxxx}$, $y_0(x) := z_0(x) - \psi(0,x) = z_0(x)$ and $y_1(x) := z_1(x) - \psi_t(0,x) = z_1(x)$ are elements of $L^2(0,T;L^2(0,L))$, $H_0^3(0,L)$ and $L^2(0,L)$ respectively, it follows that the equation

$$\begin{cases} \rho A y_{tt} + S y_{xxxx} - K y_{xxxxxx} = g, & (t,x) \in (0,T)\times(0,L), \\ y(t,0) = 0, \quad y(t,L) = 0, & t \in (0,T), \\ y_x(t,0) = 0, \quad y_x(t,L) = 0, & t \in (0,T), \\ y_{xx}(t,0) = 0, \quad y_{xx}(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), & x \in (0,L), \end{cases}$$

has a unique solution $y \in C([0,T];H_0^3(0,L)) \cap C^1([0,T];L^2(0,L))$ in virtue of Proposition 2.1. Furthermore, in view of (4) this solution satisfies

$$\|y\|_{C([0,T];H_0^3(0,L))\cap C^1([0,T];L^2(0,L))} \le C\big(\|g\|_{L^2(0,T;L^2(0,L))} + \big\|(y_0,y_1)\big\|_{H_0^3(0,L)\times L^2(0,L)}\big). \tag{12}$$

From $\psi(t,0) = \psi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \psi_{xx}(t,0) = 0$ and $\psi_{xx}(t,L) = u(t)$, we get that $z := y + \psi \in C([0,T];H_l^3(0,L)) \cap C^1([0,T];L^2(0,L))$ is a solution of Eq. (2). The inequality

$$\|\psi\|_{C([0,T];H_l^3(0,L))\cap C^1([0,T];L^2(0,L))} \le C\|u\|_{C^2([0,T])},$$

combined with $\|z\| - \|\psi\| \le \|y\|$ (valid for any norm) and (12) gives us (11). This inequality and the linearity of the equation yield the uniqueness of solutions. The proof of Proposition 2.2 is complete. □

2.2. Solutions defined by transposition

We proceed to define what we will understand as a solution defined by transposition for Eq. (1) given the data $u \in L^2(0,T)$ and $(z_0,z_1) \in L^2(0,L) \times H^{-3}(0,L)$. We proceed as in [10, Chapter 2] for instance. In order to motivate such a definition, we consider the following formal computations. For $(q_0,q_1) \in H_0^3(0,L) \times L^2(0,L)$, let $q \in C([0,T];H_0^3(0,L)) \cap C^1([0,T];L^2(0,L))$ be the unique solution of the equation

$$\begin{cases} \rho A q_{tt} + S q_{xxxx} - K q_{xxxxxx} = 0, & (t,x) \in (0,T)\times(0,L), \\ q(t,0) = 0, \quad q(t,L) = 0, & t \in (0,T), \\ q_x(t,0) = 0, \quad q_x(t,L) = 0, & t \in (0,T), \\ q_{xx}(t,0) = 0, \quad q_{xx}(t,L) = 0, & t \in (0,T), \\ q(0,x) = q_0(x), \quad q_t(0,x) = q_1(x), & x \in (0,L), \end{cases} \tag{13}$$

given by Proposition 2.1. Let $\tau \in [0,T]$. Multiplying Eq. (1) by $q = q(t,x)$ and then performing some integrations by parts on $(0,\tau)$ or $(0,L)$ we get

$$\int_0^\tau \int_0^L \left(q_{tt}(t,x) + \frac{S}{\rho A} q_{xxxx}(t,x) - \frac{K}{\rho A} q_{xxxxxx}(t,x)\right) z(t,x)\,dxdt$$
$$- \int_0^L z(t,x)q_t(t,x)\Big|_0^\tau dx + \int_0^L z_t(t,x)q(t,x)\Big|_0^\tau dx + \int_0^\tau \frac{K}{\rho A} u(t) q_{xxx}(t,L)\,dt = 0,$$

which leads us to

$$\int_0^L z(\tau,x)q_t(\tau,x)\,dx - \int_0^L z_t(\tau,x)q(\tau,x)\,dx$$

$$= \int_0^\tau \frac{K}{\rho A} u(t) q_{xxx}(t,L)\,dt + \int_0^L z_0(x) q_1(x)\,dx - \int_0^L z_1(x) q_0(x)\,dx.$$

In order to give a sense to the previous formal computations, and keeping in mind the regularity of $q = q(t,x)$ together with its hidden regularity given by Lemma 2.2, we present the following definition.

Definition 2.1. Let $(z_0, z_1) \in L^2(0,L) \times H^{-3}(0,L)$ and $u \in L^2(0,L)$. We say that $z = z(t,x)$ is a solution defined by transposition of Eq. (1) if $z \in C([0,T]; L^2(0,T)) \cap C^1([0,T]; H^{-3}(0,L))$ is such that for every $\tau \in [0,T]$ and every $(q_0, q_1) \in H_0^3(0,L) \times L^2(0,L)$ it satisfies

$$\begin{aligned}&\big\langle \big(-z_t(\tau,\cdot), z(\tau,\cdot)\big), \big(q(\tau,\cdot), q_t(\tau,\cdot)\big)\big\rangle_{H^{-3}(0,L)\times L^2(0,L), H_0^3(0,L)\times L^2(0,L)} \\ &\quad= \int_0^\tau \frac{K}{\rho A} u(t) q_{xxx}(t,L)\,dt + \int_0^L z_0(x) q_1(x)\,dx - \langle z_1, q_0\rangle_{H^{-3}(0,L), H_0^3(0,L)},\end{aligned} \tag{14}$$

with $q = q(t,x)$ being the unique solution of Eq. (13).

The next result establishes the existence and uniqueness of solutions defined by transposition for Eq. (1).

Proposition 2.3. *Let $u \in L^2(0,T)$ and $(z_0, z_1) \in L^2(0,L) \times H^{-3}(0,L)$. Then, Eq. (1) has a unique solution $z \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-3}(0,L))$ defined by transposition. Moreover, there exists $C = C(T, L, \rho, A, S, K) > 0$ such that*

$$\|z\|_{C([0,T];L^2(0,L))\cap C^1([0,T];H^{-3}(0,L))} \le C\big(\|u\|_{L^2(0,T)} + \big\|(z_0,z_1)\big\|_{L^2(0,L)\times H^{-3}(0,L)}\big). \tag{15}$$

Proof. Let us assume that $u \in \{u \in C^2([0,T]) \ / \ u(0) = u'(0) = 0\}$ and $(z_0, z_1) \in H_0^3(0,L) \times L^2(0,L)$, so that Eq. (1) has a unique solution $z \in C([0,T]; H_l^3(0,L)) \cap C^1([0,T]; L^2(0,L))$ due to Proposition 2.2. Recall that $H_l^3(0,L)$ was defined in (10) and note that in particular $z \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-3}(0,L))$.

For $(q_0, q_1) \in H_0^3(0,L) \times L^2(0,L)$, let $q \in C([0,T]; H_0^3(0,L)) \cap C^1([0,T]; L^2(0,L))$ be the unique solution of Eq. (13) given by Proposition 2.1. By taking into account that the linear map $(q(\tau,\cdot), q_t(\tau,\cdot)) \in H_0^3(0,L) \times L^2(0,L) \mapsto (q_0, q_1) \in H_0^3(0,L) \times L^2(0,L)$ is bijective thanks to Proposition 2.1, for $\tau \in [0,T]$ we define the linear form $L_\tau : H_0^3(0,L) \times L^2(0,L) \to \mathbb{R}$ given by

$$L_\tau\big(q(\tau,\cdot), q_t(\tau,\cdot)\big) := \int_0^\tau \frac{K}{\rho A} u(t) q_{xxx}(t,L)\,dt + \int_0^L z_0(x) q_1(x)\,dx - \langle z_1, q_0\rangle_{H^{-3}(0,L), H_0^3(0,L)},$$

which actually corresponds to the right-hand side of (14). This linear form is continuous in virtue of the hidden regularity stated in Lemma 2.2 and (6). In fact, from them and the Cauchy–Schwarz inequality we get

$$\begin{aligned}\big|L_\tau\big(q(\tau,\cdot), q_t(\tau,\cdot)\big)\big| &\le \frac{K}{\rho A}\|u\|_{L^2(0,\tau)}\big\|q_{xxx}(\cdot,L)\big\|_{L^2(0,\tau)} + \|z_0\|_{L^2(0,L)}\|q_1\|_{L^2(0,L)} + \|z_1\|_{H^{-3}(0,L)}\|q_0\|_{H_0^3(0,L)}, \\ &\le C\big(\|u\|_{L^2(0,T)} + \big\|(z_0,z_1)\big\|_{L^2(0,L)\times H^{-3}(0,L)}\big)\big\|(q_0,q_1)\big\|_{H_0^3(0,L)\times L^2(0,L)}, \\ &\le C\big(\|u\|_{L^2(0,T)} + \big\|(z_0,z_1)\big\|_{L^2(0,L)\times H^{-3}(0,L)}\big)\big\|\big(q(\tau,\cdot), q_t(\tau,\cdot)\big)\big\|_{H_0^3(0,L)\times L^2(0,L)}.\end{aligned} \tag{16}$$

Therefore, for every $\tau \in [0,T]$ the Riesz Representation Theorem gives us the existence of a unique $(-z_t(\tau,\cdot), z(\tau,\cdot)) \in H^{-3}(0,L) \times L^2(0,L)$ such that (14) is satisfied. Moreover, since

$$\big\|\big(z_t(\tau,\cdot), z(\tau,\cdot)\big)\big\|_{H^{-3}(0,L)\times L^2(0,L)} = \|L_\tau\|_{H^{-3}(0,L)\times L^2(0,L)},$$

it follows from (16) that

$$\big\|\big(z(\tau,\cdot),z_t(\tau,\cdot)\big)\big\|_{L^2(0,L)\times H^{-3}(0,L)}\le C\big(\|u\|_{L^2(0,T)}+\big\|(z_0,z_1)\big\|_{L^2(0,L)\times H^{-3}(0,L)}\big).$$

The previous inequality leads us to (15). By considering that the continuous injection $H_0^3(0,L)\hookrightarrow L^2(0,L)$ is dense, and hence, if we identify $L^2(0,L)$ with itself, then we obtain the same injection properties for $L^2(0,L)\hookrightarrow H^{-3}(0,L)$, we can use (15) and a density argument to conclude that Eq. (1) has a unique solution $z\in C([0,T];L^2(0,L))\cap C^1([0,T];H^{-3}(0,L))$ defined by transposition provided that $u\in L^2(0,T)$ and $(z_0,z_1)\in L^2(0,L)\times H^{-3}(0,L)$. The proof of Proposition 2.3 is complete. □

3. Boundary control

The purpose of this section is to prove Theorem 1.1, which is our main result. We begin by deriving an observability inequality for Eq. (13).

Proposition 3.1. *Let $(q_0,q_1)\in H_0^3(0,L)\times L^2(0,L)$. Then, the unique solution $q\in C([0,T];H_0^3(0,L))\cap C^1([0,T];L^2(0,L))$ of Eq. (13) satisfies*

$$\left(T-2L\max\left\{1,\frac{L^2}{\pi^2}\frac{\rho A}{S}\right\}\right)E(0)\le\frac{L}{2}\frac{K}{\rho A}\int_0^T\big|q_{xxx}(t,L)\big|^2dt. \tag{17}$$

Proof. The unique solution $q\in C([0,T];H_0^3(0,L))\cap C^1([0,T];L^2(0,L))$ for Eq. (13) is given by Proposition 2.1. The proof of this proposition is based on identity (7). In fact, from that identity, conservation of energy (5) and the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}TE(0)&\le\int_0^L xq_x(t,x)q_t(t,x)\Big|_0^T dx+\frac{L}{2}\frac{K}{\rho A}\int_0^T\big|q_{xxx}(t,L)\big|^2dt,\\&\le L\left\|\int_0^L\big(\big|q_t(t,x)\big|^2+\big|q_x(t,x)\big|^2\big)\,dx\right\|_{L^\infty(0,T)}+\frac{L}{2}\frac{K}{\rho A}\int_0^T\big|q_{xxx}(t,L)\big|^2dt.\end{aligned} \tag{18}$$

Recall that $E(t)$ was defined in (3). Since $q_x(t,\cdot)\in H_0^2(0,L)$ for every $t\in[0,T]$, the Poincaré inequality tells us that

$$\int_0^L\big|q_x(t,x)\big|^2dx\le\frac{L^2}{\pi^2}\int_0^L\big|q_{xx}(t,x)\big|^2dx.$$

Accordingly, for every $t\in[0,T]$ the above inequality allows us to obtain

$$\begin{aligned}\int_0^L\big(\big|q_t(t,x)\big|^2+\big|q_x(t,x)\big|^2\big)\,dx&\le\int_0^L\left(\big|q_t(t,x)\big|^2+\frac{L^2}{\pi^2}\frac{\rho A}{S}\frac{S}{\rho A}\big|q_{xx}(t,x)\big|^2+\frac{K}{\rho A}\big|q_{xxx}(t,x)\big|^2\right)dx,\\&\le 2\max\left\{1,\frac{L^2}{\pi^2}\frac{\rho A}{S}\right\}E(t),\end{aligned}$$

which combined with (18) and conservation of energy (5) give us the desired result. The proof of Proposition 2.2 is complete. □

In order to apply the controllability–observability duality (e.g. [6, Theorem 2.44] and [19, Theorem 11.2.1]) to prove Theorem 1.1, we need an observability inequality for the equation:

$$\begin{cases} \rho A q_{tt} + S q_{xxxx} - K q_{xxxxxx} = 0, & (t,x) \in (0,T) \times (0,L), \\ q(t,0) = 0, \quad q(t,L) = 0, & t \in (0,T), \\ q_x(t,0) = 0, \quad q_x(t,L) = 0, & t \in (0,T), \\ q_{xx}(t,0) = 0, \quad q_{xx}(t,L) = 0, & t \in (0,T), \\ q(T,x) = q_0(x), \quad q_t(T,x) = q_1(x), & x \in (0,L). \end{cases} \tag{19}$$

Note that we can transform this equation into Eq. (13) thank to the change of variable $t \mapsto T - t$. Therefore, Proposition 2.1, Lemma 2.2 and Proposition 3.1 lead us to the following result.

Proposition 3.2. *Let $(q_0, q_1) \in H_0^3(0,L) \times L^2(0,L)$. Then, Eq. (19) has a unique solution $q \in C([0,T]; H_0^3(0,L)) \cap C^1([0,T]; L^2(0,L))$ which satisfies $q_{xxx}(\cdot, L) \in L^2(0,T)$. Moreover, if we assume that*

$$T > T^* := 2L \max\left\{1, \frac{L^2}{\pi^2}\frac{\rho A}{S}\right\},$$

then there exists $C = C(T, L, \rho, A, S, K) > 0$ such that

$$\left\|(q_0, q_1)\right\|_{H_0^3(0,L)\times L^2(0,L)} \leq C\left\|q_{xxx}(\cdot, L)\right\|_{L^2(0,T)}. \tag{20}$$

We finish this section with the proof of our main result, which is the exact controllability property for control system (1).

Proof of Theorem 1.1. For $(z_0, z_1) \in L^2(0,L) \times H^{-3}(0,L)$ and $u \in L^2(0,T)$, we know that Eq. (1) has a unique solution $z \in C([0,T]; L^2(0,L)) \cap C^1([0,T]; H^{-3}(0,L))$ defined by transposition due to Proposition 2.3. Therefore, by taking into account that $(z(T,\cdot), z_t(T,\cdot)) \in L^2(0,L) \times H^{-3}(0,L)$, we introduce the set of reachable states from $(z_0, z_1) \in L^2(0,L) \times H^{-3}(0,L)$ as

$$\mathcal{R}(z_0, z_1) := \left\{\left(z(T,\cdot), z_t(T,\cdot)\right) \in L^2(0,L) \times H^{-3}(0,L) \ / \ u \in L^2(0,T)\right\}.$$

From the linearity of Eq. (1) we get

$$\mathcal{R}(z_0, z_1) = \left(\tilde{z}(T,\cdot), \tilde{z}_t(T,\cdot)\right) + \mathcal{R}(0,0),$$

where $\tilde{z} = \tilde{z}(t,x)$ is the unique solution defined by transposition of Eq. (1) when $u = 0$. Accordingly, the exact controllability property would be fulfilled if and only if $\mathcal{R}(0,0) = L^2(0,L) \times H^{-3}(0,L)$. This tells us that it is enough to study this property for the case $z_0 = z_1 = 0$. Henceforth, we may assume that $z_0 = z_1 = 0$.

Let us introduce the linear operator

$$\Lambda : u \in L^2(0,T) \mapsto \left(-z_t(T,\cdot), z(T,\cdot)\right) \in H^{-3}(0,L) \times L^2(0,L).$$

We see that the exact controllability property is equivalent to the surjectivity of operator Λ. In virtue of [4, Theorem 2.20], we have that operator Λ is surjective if and only if there exists $C > 0$ such that it holds

$$\left\|(q_0, q_1)\right\|_{H_0^3(0,L)\times L^2(0,L)} \leq C\left\|\Lambda^*(q_0, q_1)\right\|_{L^2(0,T)}, \quad \forall (q_0, q_1) \in H_0^3(0,L) \times L^2(0,L). \tag{21}$$

Let us determine adjoint operator Λ^*. For $(q_0, q_1) \in H_0^3(0,L) \times L^2(0,L)$, let $q = q(t,x)$ be the unique solution of Eq. (19) given by Proposition 3.2. Multiplying Eq. (1) by $q = q(t,x)$ and then performing some integration by parts on $(0,T)$ or $(0,L)$ we get

$$\big\langle \big(-z_t(T,\cdot), z(T,\cdot)\big), (q_0,q_1)\big\rangle_{H^{-3}(0,L)\times L^2(0,L), H_0^3(0,L)\times L^2(0,L)} = \int_0^T \frac{K}{\rho A} u(t) q_{xxx}(t,L)\, dt,$$

from where we obtain

$$\Lambda^* : (q_0,q_1) \in H_0^3(0,L) \times L^2(0,L) \mapsto \frac{K}{\rho A} q_{xxx}(t,L) \in L^2(0,T).$$

Note that adjoint operator Λ^* is well-defined thanks to Proposition 3.2. Furthermore, (20) of that proposition gives us (21), allowing us to conclude the surjectivity of operator Λ and the desired result. The proof of Theorem 1.1 is complete. □

4. Open problem

The open problem that we suggest here is inspired by the strategies followed in [7]. Let us consider the linear operator $P : H^6 \cap H_0^3(0,L) \subset L^2(0,L) \to L^2(0,L)$ defined by

$$P\phi := \frac{S}{\rho A}\frac{d^4\phi}{dx^4} - \frac{K}{\rho A}\frac{d^6\phi}{dx^6},$$

which actually corresponds to the underlying spatial operator in Eq. (1). This positive operator, that is to say that $(P\phi,\phi)_{L^2(0,L)} \geq 0$ for every $\phi \in H^6 \cap H_0^3(0,L)$ holds, is self-adjoint and its resolvent is compact. Therefore, its spectrum is a discrete set consisting only of positive eigenvalues, denoted by $\{\sigma_k\}_{k\in\mathbb{N}}$, satisfying $\lim_{k\to+\infty}\sigma_k = +\infty$. Its corresponding eigenfunctions, denoted by $\{\phi_k\}_{k\in\mathbb{N}}$, are elements of $H^6 \cap H_0^3(0,L)$ and form an orthonormal basis of $L^2(0,L)$.

The open problem that we suggest consists of two questions.

Open Problem.

1. Do the eigenvalues of operator P satisfy $\lim_{k\to+\infty}(\sqrt{\sigma_{k+1}} - \sqrt{\sigma_k}) = +\infty$?
2. Do the eigenfunctions of operator P satisfy $\phi_k'''(L) \neq 0$ for every $k \in \mathbb{N}$?

If the previous questions are positively answered, then we could use the Ingham inequality (e.g. [7, Lemma 5] and [11, Theorem 4.6]) to conclude that observability inequality (20) is valid for every $T > 0$ and not only for $T > T^*$. This would tell us that Theorem 1.1 would actually hold for $T^* = 0$.

Acknowledgments

We dedicate this work to Didier Grillon, Jean Grillon and Catherine Simonneau for their kind and warm friendship, and for making our stay in France a wonderful adventure. Patricio Guzmán has been partially supported by Anillo ACT 1106, Fondecyt 1140741 and PIIC UTFSM 2014.

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