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Lipschitz stability in an inverse problem for the main coefficient of a Kuramoto–Sivashinsky type equation

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ABSTRACT

In this paper we consider the inverse problem of retrieving the main coefficient of a Kuramoto–Sivashinsky type equation from partial knowledge of a given single solution. In particular, we focus on the stability issue of the inverse problem. By using a method due to Bukhgeim–Klibanov and suitable Carleman estimates, we obtain a local Lipschitz stability result on an admissible set of main coefficients. As a part of this study, we prove the existence of a unique solution to this equation.

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1. Introduction and main results

Let $T > 0$ and $L \in (0, +\infty)$. In this paper we consider a Kuramoto–Sivashinsky type equation, posed on Dirichlet boundary conditions, of the form

$$\begin{cases} z_t + (\sigma z_{xx})_{xx} + \gamma z_{xx} + z z_x = f, & (t, x) \in (0, T) \times (0, L), \\ z(t, 0) = h_1(t), \quad z(t, L) = h_2(t), & t \in (0, T), \\ z_x(t, 0) = h_3(t), \quad z_x(t, L) = h_4(t), & t \in (0, T), \\ z(0, x) = z_0(x), & x \in (0, L). \end{cases} \tag{1}$$

Here $\sigma = \sigma(x)$ and $\gamma = \gamma(x)$ are known as the diffusion and anti-diffusion coefficients respectively, and $f = f(t, x)$ is an external excitation to the system. Eq. (1), with σ and γ being positive constants, is the Kuramoto–Sivashinsky equation which was derived independently by Kuramoto and Tsuzuki [15] as a model for phase turbulence in reaction–diffusion systems, and by Sivashinsky [20] as a model for the physical phenomena of plane flame propagation. This equation also arises in the modeling of the flow of a thin film of viscous liquid falling down on an inclined plane subject to an applied electric field [10].

Assuming that γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$ are known and fixed, we consider the inverse problem of retrieving σ, which is the main coefficient of Eq. (1), from partial knowledge of a given single solution. Some of the required knowledge comes from measurements of the solution at the boundary, and since this equation is parabolic, our method also requires knowledge of the solution at some positive time in $(0, T)$. This last requirement is characteristic of inverse problems for parabolic equations (e.g. [1,3,12,22]) and has already been discussed in [12]. Being more precise, we study the following.

Inverse problem: Retrieve σ from the measurement of $z_{xx}(\cdot, 0)$ and $z_{xxx}(\cdot, 0)$ in $(0, T)$, and from the measurement of $z(T_0, \cdot)$ in $(0, L)$, where z is the solution of Eq. (1) and $T_0 \in (0, T)$.

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By "retrieve" in an inverse problem we mean to obtain uniqueness, stability and reconstruction results. However, in this paper we focus on the stability issue. Our main result is a local Lipschitz stability on an admissible set of main coefficients. Of course, local uniqueness is also achieved on the same set.

The admissible set of main coefficients that we consider is defined by

$$\Sigma_{ad} := \Sigma_{ad}(\sigma_0, \varepsilon_1, \varepsilon_2, m_1, m_2) := \{\sigma \in H^4(0, L)/\sigma(x) \geq \sigma_0 \ \forall x \in (0, L), \sigma(0) = \varepsilon_1, \sigma_x(0) = \varepsilon_2, \|\sigma\|_{H^4(0,L)} \leq m_1, z(\sigma) \in H^1(0, T; H^4(0, L)), \|z(\sigma)\|_{W^{1,\infty}(0,T;W^{1,\infty}(0,L))} \leq m_2\}, \tag{2}$$

where the constants $0 < \sigma_0 \leq \varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $m_1, m_2 > 0$ are given; and $z(\sigma)$ denotes the solution of Eq. (1) associated to σ, γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$. The non-emptiness of Σ_{ad} is an issue to be studied and is related to the existence of solutions of Eq. (1). This kind of result has already been obtained in [1], however, it only holds for data $(f, h_1, h_2, h_3, h_4, z_0)$ sufficiently small, i.e.

$$\|f\| + \sum_{i=1}^{4} \|h_i\| + \|z_0\| \leq \varepsilon, \tag{3}$$

(with appropriate norms) for $\varepsilon > 0$ small enough. In this paper we succeeded to remove assumption (3), for a slightly less regular functional framework than the presented in [1], by obtaining suitable energy estimates for the solution of Eq. (1). Although the price to pay is to obtain less regular solutions, they are regular enough for our purpose.

In order to present our result concerning the existence of a unique solution of Eq. (1), some definitions will be needed. Consider the auxiliary function

$$\psi(t, x) := h_1(t)d_1(x) + h_2(t)d_2(x) + h_3(t)d_3(x) + h_4(t)d_4(x), \tag{4}$$

with $d_1(x) := 2L^{-3}x^3 - 3L^{-2}x^2 + 1$, $d_2(x) := -2L^{-3}x^3 + 3L^{-2}x^2$, $d_3(x) := L^{-2}x^3 - 2L^{-1}x^2 + x$ and $d_4(x) := L^{-2}x^3 - L^{-1}x^2$. With this choice for these polynomials we get $\psi(t, 0) = h_1(t)$, $\psi(t, L) = h_2(t)$, $\psi_x(t, 0) = h_3(t)$ and $\psi_x(t, L) = h_4(t)$. This allows us to present the following definition.

Definition 1. Suppose that z satisfies Eq. (1). Let ψ be the auxiliary function defined in (4). σ, γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$ are compatible if $z(0, x) = \psi(0, x)$, $z_x(0, x) = \psi_x(0, x)$, $z_t(0, x) = \psi_t(0, x)$ and $z_{tx}(0, x) = \psi_{tx}(0, x)$ hold at $x = 0$ and $x = L$.

The first two equalities in Definition 1 are relations between z_0 and (h_1, h_2, h_3, h_4), and are required to obtain regular solutions z of Eq. (1) in $C([0, T]; H^2(0, L)) \cap L^2(0, T; H^4(0, L))$; whereas the last two, are relations between σ, γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$, and are required to get further regularity for z, namely, $z_t \in C([0, T]; H^2(0, L)) \cap L^2(0, T; H^4(0, L))$.

Theorem 1. *Let $\sigma_0 > 0$ and $\sigma \in H^4(0, L)$ be such that $\sigma(x) \geq \sigma_0 > 0$ for every $x \in (0, L)$. Let $\gamma \in H^1(0, T; H^2(0, L))$, $f \in C([0, T]; H^2(0, L)) \cap H^1(0, T; L^2(0, L))$, $(h_1, h_2, h_3, h_4) \in H^2(0, T)^4$ and $z_0 \in H^6(0, L)$. Assume that they are compatible. Then, Eq. (1) with $\gamma = \gamma(t, x)$ has a unique solution $z \in C([0, T]; H^6(0, L))$ with $z_t \in C([0, T]; H^2(0, L)) \cap L^2(0, T; H^4(0, L))$. Moreover, there exists $C = C(L, T, \|\sigma\|_{H^4(0,L)}, \sigma_0, \|\gamma\|_{H^1(0,T;H^2(0,L))}) > 0$ such that*

$$\|z\|_{C([0,T];H^6(0,L))} + \|z_t\|_{C([0,T];H^2(0,L))\cap L^2(0,T;H^4(0,L))} \leq \exp\left\{\exp\left\{C\left(1 + \|f\|_{C([0,T];H^2(0,L))\cap H^1(0,T;L^2(0,L))} + \|(h_1, h_2, h_3, h_3)\|_{H^2(0,T)^4} + \|z_0\|_{H^6(0,L)}\right)\right\}\right\}. \tag{5}$$

Remark 1. For the existence of a unique solution z of Eq. (1) we use (16), where an exponential is involved. A similar estimate is obtained, for instance, in Lemma 21 in [7] for the Korteweg–de Vries equation. The double exponential in (5) appears when (16) is applied to z_t in order to obtain more regularity for z.

As it was pointed out before, this result allows us to argue that Σ_{ad} is a non-empty set in some cases. In fact, provided that σ, γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$ are regular enough and compatible, we see that if we set m_2^- as the right-hand side of (5), then we have that Σ_{ad} is a non-empty set if $m_2 \in [m_2^-, \infty)$.

Now we can present our main result.

Theorem 2. *Consider σ, γ and data $(f, h_1, h_2, h_3, h_4, z_0)$ regular enough and compatible. Assume that there exist $\overline{\sigma} \in \Sigma_{ad}$, $T_0 \in (0, T)$ and $R_0 > 0$ such that $|z(\overline{\sigma})_{xx}(T_0, x)| \geq R_0$ for every $x \in (0, L)$. Then, there exists $C = C(m_1, m_2, L, T, \sigma_0, R_0, \|\gamma\|_{L^\infty(0,L)}) > 0$ such that for every $\sigma \in \Sigma_{ad}$ we have*

$$\frac{1}{C}\|\sigma - \overline{\sigma}\|_{H^2(0,L)} \leq \|z(\sigma)(T_0, \cdot) - z(\overline{\sigma})(T_0, \cdot)\|_{H^4(0,L)} + \|z_{xx}(\sigma)(\cdot, 0) - z_{xx}(\overline{\sigma})(\cdot, 0)\|_{H^1(0,T)} + \|z_{xxx}(\sigma)(\cdot, 0) - z_{xxx}(\overline{\sigma})(\cdot, 0)\|_{H^1(0,T)} \leq C\|z(\sigma) - z(\overline{\sigma})\|_{H^1(0,T;H^4(0,L))}. \tag{6}$$

The first inequality in (6) corresponds to the local Lipschitz stability result for our inverse problem, and the second one, to the regularity of the measurements. Indeed, the second inequality indicates that the required measurements are finite if $\sigma, \overline{\sigma} \in \Sigma_{ad}$.

To prove Theorem 2 we use a method due to Bukhgeim–Klibanov [4] and suitable Carleman estimates, which are families of weighted energy estimates for the solution of a partial differential equation. The suitable Carleman estimates needed are two: one for fourth order parabolic operators and another one for second order elliptic operators. This is due to the fact that Eq. (1) without the nonlinear term zz_x is a fourth order parabolic equation, and if we consider Eq. (1) at $t = T_0$, then we obtain that σ is the main coefficient of a second order elliptic equation.

Remark 2. In Theorem 2 the knowledge that $z_{xx}(\overline{\sigma})$, which is called reference trajectory, at $t = T_0$ does not change its sign in $(0, L)$ seems to be the price to pay to retrieve the coefficient σ with only one measurement. This kind of assumption is usual when the method of Bukhgeim–Klibanov is applied in inverse problems for parabolic equations (e.g. [1,3,12,22]). The existence or not of such a trajectory is an open problem.

Remark 3. The analogousness in inverse problems for hyperbolic equations (e.g. [8,13,17]), or equations with similar properties such as the Schrödinger equation (e.g. [2,16,23]), is to know a reference trajectory that at $t = 0$ does not change its sign in $(0, L)$.

To the knowledge of the author, there are two previous papers proving Carleman estimates for fourth order parabolic operators with the aim to be used in the following contexts:

- *Control theory.* In [6] local exact controllability to the trajectories is obtained for the Kuramoto–Sivashinsky equation.
- *Inverse problems.* In [1] a local Lipschitz stability result is obtained for the inverse problem of retrieving the anti-diffusion coefficient γ of a Kuramoto–Sivashinsky type equation.

Those estimates only have one large positive parameter. The Carleman estimate obtained in [6] is not useful for our purpose because it only applies for operators with constant coefficients. The Carleman estimate obtained in [1] might be suitable to obtain a stability result for our inverse problem, however its weight functions depends on every σ chosen. To deal with this difficulty we derive a new one by using others weight functions, none of them depending on σ, and by incorporating a second large positive parameter. This estimate is presented in Proposition 3 and is obtained by following a procedure due to A.V. Fursikov and O.Y. Imanuvilov [9]. For a review about Carleman estimates for parabolic operators you can see [21].

There already exist Carleman estimates for second order elliptic operators for general functional frameworks as can be seen, for instance, in [11,14,19]. However, we need one in such a way that it can be used together with our Carleman estimate for fourth order parabolic operators. Such an estimate is presented in Proposition 5 and its weight functions are the same as the first estimate but evaluated at $t = T_0$.

For other papers obtaining local Lipschitz stability results for inverse problems of retrieving the main coefficient of parabolic equations we mention [3,22], where it is considered the heat equation with discontinuous main coefficient and second order parabolic equations respectively.

This paper is organized as follows. In Section 2 we prove the existence of a unique solution for the Kuramoto–Sivashinsky type equation being considered. In Section 3 we prove the suitable Carleman estimates needed. Finally, in Section 4 we prove the local Lipschitz stability result for our inverse problem.

2. Well-posedness

In this section we study the well-posedness of Eq. (1). The main tools to use are the Semigroup Theory, suitable energy estimates and the Banach Fixed Point Theorem.

2.1. Linear problem

We consider the simplest linear equation, with null boundary conditions, associated to Eq. (1) such that we can use the Semigroup Theory.

Proposition 1. *Let $\sigma_0 > 0$ and $\sigma \in H^2(0, L)$ be such that $\sigma(x) \geq \sigma_0$ for every $x \in (0, L)$. Let $f \in L^2(0, T; L^2(0, L))$ and $z_0 \in H_0^2(0, L)$. Then, the equation*

$$\begin{cases} z_t + (\sigma z_{xx})_{xx} = f, & (t, x) \in (0, T) \times (0, L), \\ z(t, 0) = 0, \quad z(t, L) = 0, & t \in (0, T), \\ z_x(t, 0) = 0, \quad z_x(t, L) = 0, & t \in (0, T), \\ z(0, x) = z_0(x), & x \in (0, L), \end{cases} \tag{7}$$

has a unique solution $z \in C([0, T]; H_0^2(0, L)) \cap L^2(0, T; H^4(0, L))$. Moreover, there exists $C = C(L, \|\sigma\|_{H^2(0,L)}, \sigma_0) > 0$ such that

$$\|z\|_{C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L))} \leq C\sqrt{1 + T^2}\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{H^2(0,L)}\right). \tag{8}$$

Proof. The proof of this proposition is split into two parts.

Part 1: *Semigroup Theory.* Consider the linear operator $A : H^4 \cap H_0^2(0, L) \subset L^2(0, L) \to L^2(0, L)$ defined by $Az := -(\sigma z_{xx})_{xx}$. This operator allows us to write Eq. (7) into the Cauchy problem

$$\begin{cases} z'(t) = Az(t) + f(t), & t \in [0, T], \\ z(0) = z_0. \end{cases} \tag{9}$$

We have that A is a self-adjoint operator and, since $\sigma(x) \geq \sigma_0 > 0$ for every in $x \in (0, L)$, we also obtain that A is dissipative, i.e. $(Az, z)_{L^2(0,L)} \leq 0$ for all $z \in H^4 \cap H_0^2(0, L)$. Therefore, A is a m-dissipative operator (e.g. Corollary 2.4.8 in [5]) and by the Hille–Yosida–Phillips theorem (e.g. Theorem 3.4.4 in [5]) we obtain that A is a generator of a contraction semigroup in $L^2(0, L)$. Thus, if $z_0 \in H^4 \cap H_0^2(0, L)$ and $f \in C^1([0, T]; L^2(0, L))$, then Eq. (9) has solutions $z \in C([0, T]; H^4 \cap H_0^2(0, L)) \cap C^1([0, T]; L^2(0, L))$ (e.g. Proposition 4.1.6 in [5]).

Part 2: *Energy estimates.* Throughout this section the letter C will denote a positive constant which may vary from line to line. For the moment we assume that $z_0 \in H^4 \cap H_0^2(0, L)$ and $f \in C^1([0, T]; L^2(0, L))$.

Multiplying by $z(t, x)$ to Eq. (7) and then performing integration by parts over $(0, L)$ we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |z(t,x)|^2 dx\right) + \int_0^L \sigma(x)|z_{xx}(t,x)|^2 dx = \int_0^L f(t,x)z(t,x)\, dx.$$

Since from this equality we have

$$\|z(t,\cdot)\|^2_{L^2(0,L)} + 2\sigma_0\|z_{xx}\|^2_{L^2(0,T;L^2(0,L))} \leq 4\int_0^T \int_0^L f(t,x)z(t,x)\, dxdt + 2\|z_0\|^2_{L^2(0,L)},$$

we can use the inequality $4Tab \leq 8T^2a^2 + (1/2)b^2$, with $a = f(t, x)$ and $b = z(t, x)$, on it to obtain

$$\|z\|^2_{L^2(0,T;L^2(0,L))} + 2\sigma_0 T\|z_{xx}\|^2_{L^2(0,T;L^2(0,L))} \leq 8T^2\|f\|^2_{L^2(0,T;L^2(0,L))} + 2T\|z_0\|^2_{L^2(0,L)}. \tag{10}$$

Multiplying by $(\sigma(x)z_{xx}(t, x))_{xx}$ to Eq. (7) and then performing integration by parts over $(0, L)$ we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L \sigma(x)|z_{xx}(t,x)|^2 dx\right) + \int_0^L |(\sigma(x)z_{xx}(t,x))_{xx}|^2 dx = \int_0^L f(t,x)(\sigma(x)z_{xx}(t,x))_{xx}\, dx, \tag{11}$$

from which we can conclude

$$\sigma_0\|z_{xx}(t,\cdot)\|^2_{L^2(0,L)} + \|(\sigma z_{xx})_{xx}\|^2_{L^2(0,T;L^2(0,L))} \leq 2\|f\|^2_{L^2(0,T;L^2(0,L))} + 2\|\sigma\|_{L^\infty(0,L)}\|z_0\|^2_{H^2(0,L)}. \tag{12}$$

The continuous injection $H^1(0, L) \hookrightarrow L^\infty(0, L)$ allows us to obtain

$$\|z\|^2_{L^2(0,T;H^4(0,L))} \leq C\left(\|(\sigma z_{xx})_{xx}\|^2_{L^2(0,T;L^2(0,L))} + \|z\|^2_{L^2(0,T;H^3(0,L))}\right). \tag{13}$$

To handle the last term of right-hand side of (13) we use a special case of Ehrling's lemma (e.g. Theorem 7.30 in [18]): for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for every $z \in L^2(0, T; H^4(0, L))$ we have

$$\|z\|^2_{L^2(0,T;H^3(0,L))} \leq \varepsilon\|z\|^2_{L^2(0,T;H^4(0,L))} + C(\varepsilon)\|z\|^2_{L^2(0,T;L^2(0,L))}. \tag{14}$$

Plugging (10) in (14), with $\varepsilon = 1/2C$, and then combining it with (13) and (12) gives us

$$\|z_{xx}(t,\cdot)\|^2_{L^2(0,L)} + \|z\|^2_{L^2(0,T;H^4(0,L))} \leq C(1+T^2)\left(\|f\|^2_{L^2(0,T;L^2(0,L))} + \|z_0\|^2_{H^2(0,L)}\right).$$

From this inequality and the Poincaré inequality we arrive to (8), which allows us to use a density argument to conclude that Eq. (7) has a unique solution $z \in C([0, T]; H_0^2(0, L)) \cap L^2(0, T; H^4(0, L))$ if $f \in L^2(0, T; L^2(0, L))$ and $z_0 \in H_0^2(0, L)$. The proof of Proposition 1 is complete. □

2.2. Nonlinear problem

We begin this section deriving an a priori estimate in $C([0, T]; H_0^2(0, L)) \cap L^2(0, T; H^4(0, L))$ for any solution of Eq. (1) with $\gamma = \gamma(t, x)$, low order terms and null boundary conditions.

Lemma 1. *Let $\sigma_0 > 0$ and $\sigma \in H^2(0, L)$ be such that $\sigma(x) \geq \sigma_0$ for every $x \in (0, L)$. Let $\gamma \in L^\infty(0, T; L^\infty(0, L))$, $\beta \in L^\infty(0, T; H^2(0, L))$, $\alpha \in \mathbb{R}$, $f \in L^2(0, T; L^2(0, L))$ and $y_0 \in H_0^2(0, L)$. Consider the equation*

$$\begin{cases} y_t + (\sigma y_{xx})_{xx} + \gamma y_{xx} + \beta y_x + \beta_x y + \alpha y y_x = f, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0, \quad y(t,L) = 0, & t \in (0,T), \\ y_x(t,0) = 0, \quad y_x(t,L) = 0, & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,L). \end{cases} \tag{15}$$

Then, there exists $C = C(L, T, \|\sigma\|_{H^2(0,L)}, \sigma_0, \|\gamma\|_{L^\infty(0,T;L^\infty(0,L))}) > 0$ such that the following a priori estimate holds

$$\|y\|_{C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L))} \le \exp\left\{C(1+T)\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right) + |\alpha| \exp\left\{C(1+T)\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|f\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{H^2(0,L)}\right). \tag{16}$$

Proof. Assume that Eq. (15) has solutions regular enough to perform the following computations. We start by deriving an a priori estimate in $C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^2(0,L))$ for any y satisfying (15). This will lead us to an a priori estimate in $C([0,T]; H_0^2(0,L)) \cap L^2(0,T; H^4(0,L))$.

Multiplying by $y(t,x)$ to system (15) and then performing integration by parts over $(0,L)$ we get

$$\frac{1}{2}\frac{d}{dt}\left(\int_0^L |y(t,x)|^2 dx\right) + \int_0^L \sigma(x)|y_{xx}(t,x)|^2 dx + \int_0^L \gamma(x) y_{xx}(t,x) y(t,x)\, dx + \frac{1}{2}\int_0^L \beta_x(t,x)|y(t,x)|^2 dx = \int_0^L f(t,x) y(t,x)\, dx.$$

If in this equality we consider

$$\int_0^L \gamma(x) y_{xx}(t,x) y(t,x)\, dx \le \frac{\|\gamma\|^2_{L^\infty(0,T;L^\infty(0,L))}}{2\sigma_0}\int_0^L |y(t,x)|^2 dx + \frac{1}{2}\int_0^L \sigma(x)|y_{xx}(t,x)|^2 dx,$$

then by using the continuous injection $H^1(0,L) \hookrightarrow L^\infty(0,L)$ we arrive to

$$\frac{d}{dt}\left(\int_0^L |y(t,x)|^2 dx\right) + \int_0^L \sigma(x)|y_{xx}(t,x)|^2 dx \le C\left(1 + \|\beta(t,\cdot)\|_{H^2(0,L)}\right)\int_0^L |y(t,x)|^2 dx + \int_0^L |f(t,x)|^2 dx.$$

Grönwall's lemma allows us to conclude from this inequality that

$$\|y(t,\cdot)\|^2_{L^2(0,L)} \le \exp\left\{C\left(T + \|\beta\|_{L^1(0,T;H^2(0,L))}\right)\right\}\left(\|f\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{L^2(0,L)}\right). \tag{17}$$

Moreover, according to the Poincaré inequality, (17) and the inequality $x \le e^x$ for all $x \in \mathbb{R}$ we obtain

$$\|y\|^2_{L^2(0,T;H_0^2(0,L))} \le C\left(T + \|\beta\|_{L^1(0,T;H^2(0,L))}\right)\|y\|^2_{C([0,T];L^2(0,L))} + \|f\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{L^2(0,L)},$$

$$\le \exp\left\{C\left(1 + T + \|\beta\|_{L^1(0,T;H^2(0,L))}\right)\right\}\left(\|f\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{L^2(0,L)}\right). \tag{18}$$

Therefore, combining (17) and (18) leads us to

$$\|y\|_{C([0,T];L^2(0,L))\cap L^2(0,T;H_0^2(0,L))} \le \exp\left\{C(1+T)\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)}\right). \tag{19}$$

Applying (8) to Eq. (15) we get

$$\|y\|_{C([0,T];H_0^2(0,L))\cap L^2(0,T;H^4(0,L))} \le C\sqrt{1+T^2}\left(\|f - \gamma y_{xx} - \beta y_x - \beta_x y - \alpha y y_x\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right). \tag{20}$$

The first term of the right-hand side of (20) will be bounded from above in terms of $\|y\|_{C([0,T];L^2(0,L))}$ and $\|y\|_{L^2(0,T;H^2(0,L))}$ only. The continuous injection $H^1(0,L) \hookrightarrow L^\infty(0,L)$ will be used several times in order to achieve this. The following inequalities are obtained:

$$\|\gamma y_{xx}\|_{L^2(0,T;L^2(0,L))} \le \|\gamma\|_{L^\infty(0,T;L^\infty(0,L))}\|y\|_{L^2(0,T;H^2(0,L))},$$

$$\|\beta y_x\|_{L^2(0,T;L^2(0,L))} \le \left(\int_0^T \|\beta(t,\cdot)\|^2_{L^\infty(0,L)}\|y_x(t,\cdot)\|^2_{L^2(0,L)} dt\right)^{1/2} \le C\|\beta\|_{L^\infty(0,T;H^1(0,L))}\|y\|_{L^2(0,T;H^1(0,L))},$$

$$\|\beta_x y\|_{L^2(0,T;L^2(0,L))} \le \left(\int_0^T \|\beta_x(t,\cdot)\|^2_{L^\infty(0,L)}\|y(t,\cdot)\|^2_{L^2(0,L)} dt\right)^{1/2} \le C\|\beta\|_{L^\infty(0,T;H^2(0,L))}\|y\|_{L^2(0,T;L^2(0,L))},$$

$$\|\alpha y y_x\|_{L^2(0,T;L^2(0,L))} \le |\alpha|\left(\int_0^T \|y(t,\cdot)\|^2_{L^2(0,L)}\|y_x(t,\cdot)\|^2_{L^\infty(0,L)}\right)^{1/2} \le C|\alpha|\,\|y\|_{C([0,T];L^2(0,L))}\|y\|_{L^2(0,T;H^2(0,L))}.$$

Hence, plugging (19) on these inequalities gives us

$$\begin{aligned}\|\gamma y_{xx} + \beta y_x + \beta_x y\|_{L^2(0,T;L^2(0,L))} &\leq C\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\|y\|_{L^2(0,T;H^2(0,L))},\\ &\leq \exp\left\{C\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{L^2(0,L)}\right),\\ \|\alpha y y_x\|_{L^2(0,T;L^2(0,L))} &\leq |\alpha|\|y\|_{C([0,T];L^2(0,L))}\|y\|_{L^2(0,T;H^2(0,L))},\\ &\leq |\alpha|\exp\left\{C\left(1 + \|\beta\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|f\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{L^2(0,L)}\right).\end{aligned}$$

Finally, (16) is obtained by combining the two preceding inequalities with (20) and then using the inequality $x \leq e^x$ for all $x \in \mathbb{R}$. The proof of Lemma 1 is complete. □

It will be shown that Eq. (1) with $\gamma = \gamma(t,x)$ has a unique solution in $C([0,T];H^2(0,L)) \cap L^2(0,T;H^4(0,L))$: the Banach Fixed Point Theorem will give us the local well-posedness and Lemma 1 the global well-posedness.

Proposition 2. *Let $\sigma_0 > 0$ and $\sigma \in H^2(0,L)$ be such that $\sigma(x) \geq \sigma_0$ for every $x \in (0,L)$. Let $\gamma \in L^\infty(0,T;L^\infty(0,L))$, $f \in L^2(0,T;L^2(0,L))$, $(h_1,h_2,h_3,h_4) \in H^1(0,T)^4$ and $z_0 \in H^2(0,L)$. Assume that*

$$z_0(0) = h_1(0), \qquad z_0(L) = h_2(0), \qquad z_0'(0) = h_3(0) \quad \text{and} \quad z_0'(L) = h_4(0). \tag{21}$$

Then, Eq. (1) with $\gamma = \gamma(t,x)$ has a unique solution $z \in C([0,T];H^2(0,L)) \cap L^2(0,T;H^4(0,L))$. Moreover, there exists $C = C\left(L,T,\|\sigma\|_{H^2(0,L)},\sigma_0,\|\gamma\|_{L^\infty(0,T;L^\infty(0,L))}\right) > 0$ such that

$$\|z\|_{C([0,T];H^2(0,L))\cap L^2(0,T;H^4(0,L))} \leq \exp\left\{C\left(1 + \|f\|_{L^2(0,T;L^2(0,L))} + \|(h_1,h_2,h_3,h_4)\|_{H^1(0,T)^4} + \|z_0\|_{H^2(0,L)}\right)\right\}. \tag{22}$$

Before giving the proof of this proposition, we remark that to ask (21) corresponds to ask that $z(0,x) = \psi(0,x)$ and $z_x(0,x) = \psi_x(0,x)$ hold at $x = 0$ and $x = L$ (see Definition 1). Recall that ψ is the auxiliary function defined in (4).

Proof of Proposition 2. Assume that z satisfies Eq. (1) with $\gamma = \gamma(t,x)$. Let us define $y := z - \psi$. From Eq. (1) it follows that y satisfies

$$\begin{cases} y_t + (\sigma y_{xx})_{xx} + \gamma y_{xx} + \psi y_x + \psi_x y + y y_x = g, & (t,x) \in (0,T)\times(0,L),\\ y(t,0) = 0, \qquad y(t,L) = 0, & t \in (0,T),\\ y_x(t,0) = 0, \qquad y_x(t,L) = 0, & t \in (0,T),\\ y(0,x) = y_0(x), & x \in (0,L),\end{cases} \tag{23}$$

where $g := f - \psi_t - (\sigma\psi_{xx})_{xx} - \gamma\psi_{xx} - \psi\psi_x$ and $y_0(x) := z_0(x) - \psi(0,x)$. We have that $\psi \in H^1(0,T;C^\infty([0,L]))$, and from the inequalities

$$\|\psi\|_{L^\infty(0,T;H^2(0,L))} \leq C\|(h_1,h_2,h_3,h_4)\|_{L^\infty(0,T)^4}, \tag{24}$$

$$\|g\|_{L^2(0,T;L^2(0,L))} \leq C\left(\|f\|_{L^2(0,T;L^2(0,L))} + \|(h_1,h_2,h_3,h_4)\|_{H^1(0,T)^4} + \|(h_1,h_2,h_3,h_4)\|^2_{L^\infty(0,T)^4}\right), \tag{25}$$

$$\|y_0\|_{H^2(0,L)} \leq C\left(\|z_0\|_{H^2(0,L)} + \|(h_1,h_2,h_3,h_4)\|_{L^\infty(0,T)^4}\right), \tag{26}$$

that $g \in L^2(0,T;L^2(0,L))$. Moreover, $y_0 \in H_0^2(0,L)$ is obtained by additionally considering (21). The remainder of the proof of this proposition is split into three parts.

Part 1: *Uniqueness.* Suppose that Eq. (23) has a unique solution $y \in C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$, and consider $\bar{z} \in C([0,T];H^2(0,L)) \cap L^2(0,T;H^4(0,L))$ to be any solution of Eq. (1) with $\gamma = \gamma(t,x)$. We can use Lemma 1, with $\beta = z = y + \psi$ and $\alpha = 0$, on the equation satisfied by $\Delta := z - \bar{z}$ to conclude that $\Delta = 0$, leading us to the uniqueness of solutions.

Part 2: *Local well-posedness.* For $\theta \in (0,T]$ consider $\mathcal{F}_\theta := C([0,\theta];H_0^2(0,L)) \cap L^2(0,\theta;H^4(0,L))$, and for $R > 0$, $B_{\theta,R} := \{z \in \mathcal{F}_\theta / \|z\|_{\mathcal{F}_\theta} \leq R\}$, which is a Banach space endowed with the $\|\cdot\|_{\mathcal{F}_\theta}$ norm.

Let us define the map $\Lambda : p \in B_{\theta,R} \mapsto \Lambda(p) = y \in B_{\theta,R}$, where y satisfies

$$\begin{cases} y_t + (\sigma y_{xx})_{xx} = g - (\gamma p_{xx} + \psi p_x + \psi_x p + p p_x), & (t,x) \in (0,\theta)\times(0,L),\\ y(t,0) = 0, \qquad y(t,L) = 0, & t \in (0,\theta),\\ y_x(t,0) = 0, \qquad y_x(t,L) = 0, & t \in (0,\theta),\\ y(0,x) = y_0(x), & x \in (0,L).\end{cases} \tag{27}$$

In this part of the proof it will be shown that Λ is well defined and a contraction for $\theta \in (0,T)$ small enough and for a certain $R > 0$, that later will be chosen properly. Note that p is a fixed point of Λ if and only if p is a solution of Eq. (23) with $T = \theta$.

From the inequalities

$$\|\gamma p_{xx} + \psi p_x + \psi_x p\|_{L^2(0,\theta;L^2(0,L))} \le \left(\|\gamma\|_{L^\infty(0,L)} + \|\psi\|_{L^\infty(0,\theta;W^{1,\infty}(0,L))}\right)\theta^{1/2}\|p\|_{\mathcal{F}_\theta},$$
$$\|pp_x\|_{L^2(0,\theta;L^2(0,L))} \le \theta^{1/2}\|p\|^2_{C([0,\theta];H^1(0,L))} \le \theta^{1/2}\|p\|^2_{\mathcal{F}_\theta},$$

we have that $(g - \gamma p_{xx} - \psi p_x - \psi_x p - pp_x) \in L^2(0,\theta;L^2(0,L))$. Hence, we can use Proposition 1 to conclude that Eq. (27) has a unique solution $\Lambda(p) = y \in \mathcal{F}_\theta$. Moreover, in virtue of (8), that solution satisfies

$$\|\Lambda(p)\|_{\mathcal{F}_\theta} \le C\sqrt{1+\theta^2}\left(\|g\|_{L^2(0,\theta;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right)$$
$$+ C\sqrt{1+\theta^2}\left(\|\gamma\|_{L^\infty(0,L)} + \|\psi\|_{L^\infty(0,\theta;W^{1,\infty}(0,L))} + R\right)\theta^{1/2}\|p\|_{\mathcal{F}_\theta}. \tag{28}$$

Let $p, q \in B_{\theta,R}$. From Eq. (27) it follows that $\Delta := \Lambda(p) - \Lambda(q)$ satisfies

$$\begin{cases} \Delta_t + (\sigma\Delta_{xx})_{xx} = \gamma(q_{xx} - p_{xx}) + \psi(q_x - p_x) + \psi_x(q - p) + (qq_x - pp_x), & (t,x) \in (0,\theta)\times(0,L),\\ \Delta(t,0) = 0, \qquad \Delta(t,L) = 0, & t \in (0,\theta),\\ \Delta_x(t,0) = 0, \qquad \Delta_x(t,L) = 0, & t \in (0,\theta),\\ \Delta(0,x) = 0, & x \in (0,L). \end{cases}$$

Applying (8) on this equation and then combining the resulting expression with the inequalities

$$\|pp_x - qq_x\|_{L^2(0,\theta;L^2(0,L))} \le \|(p-q)p_x\|_{L^2(0,\theta;L^2(0,L))} + \|(p_x - q_x)q\|_{L^2(0,\theta;L^2(0,L))},$$
$$\|p_x\|_{L^\infty(0,\theta;L^\infty(0,L))} \le R \quad \text{and} \quad \|q\|_{L^\infty(0,\theta;L^\infty(0,L))} \le R,$$

give us

$$\|\Delta\|_{\mathcal{F}_\theta} \le C\sqrt{1+\theta^2}\left(\|\gamma\|_{L^\infty(0,L)} + \|\psi\|_{L^\infty(0,\theta;W^{1,\infty}(0,L))} + 2R\right)\theta^{1/2}\|p - q\|_{\mathcal{F}_\theta}. \tag{29}$$

Hence, if we define

$$\begin{cases} R := 2C\sqrt{1+\theta^2}\left(\|g\|_{L^2(0,\theta;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right),\\ M(\theta) := C\sqrt{1+\theta^2}\left(\|\gamma\|_{L^\infty(0,L)} + \|\psi\|_{L^\infty(0,\theta;W^{1,\infty}(0,L))} + 2R\right)\theta^{1/2}, \end{cases}$$

then from (28) and (29) we obtain

$$\begin{cases} \|\Lambda(p)\|_{\mathcal{F}_\theta} \le \dfrac{R}{2} + M(\theta)\|p\|_{\mathcal{F}_\theta},\\ \|\Lambda(p) - \Lambda(q)\|_{\mathcal{F}_\theta} \le M(\theta)\|p - q\|_{\mathcal{F}_\theta}. \end{cases} \tag{30}$$

Since $M(0^+) = 0$ and $M(\cdot)$ is a non-negative, continuous and increasing function on $(0,+\infty)$, there exists $\theta \in (0,T)$ small enough such that $M(\theta) \le 1/2$. Therefore, from (30) we have that Λ is well defined and a contraction. Finally, the Banach Fixed Point Theorem allows us to conclude that Λ has a unique fixed point in $B_{\theta,R}$.

Part 3: *Global well-posedness.* Since we already have a local result for Eq. (23), we only need to obtain an a priori estimate for any of its solutions to achieve a global result.

From the beginning of the proof of this proposition we see that Lemma 1 can be used, with $\beta = \psi$ and $\alpha = 1$, on Eq. (23) to obtain

$$\|y\|_{\mathcal{F}_T} \le \exp\left\{C(1+T)\left(1 + \|\psi\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right)$$
$$+ \exp\left\{C(1+T)\left(1 + \|\psi\|_{L^\infty(0,T;H^2(0,L))}\right)\right\}\left(\|g\|^2_{L^2(0,T;L^2(0,L))} + \|y_0\|^2_{H^2(0,L)}\right).$$

Therefore, plugging (24) to (26) on this inequality and then using the inequality $x \le e^x$ for all $x \in \mathbb{R}$ give us

$$\|y\|_{\mathcal{F}_T} \le \exp\left\{C\left(1 + \|f\|_{L^2(0,T;L^2(0,L))} + \|(h_1,h_2,h_3,h_4)\|_{H^1(0,T)^4} + \|z_0\|_{H^2(0,L)}\right)\right\},$$

from which we get (22), and the global result, by taking into account $\|z\|_{\mathcal{F}_T} - \|\psi\|_{\mathcal{F}_T} \le \|y\|_{\mathcal{F}_T}$. The proof of Proposition 2 is complete. □

The estimates developed so far will be used to obtain more regularity for the solution of Eq. (1) with $\gamma = \gamma(t,x)$. Further regularity requires to ask for a compatibility condition on σ, γ and the data $(f, h_1, h_2, h_3, h_4, z_0)$. This compatibility condition is given by Definition 1.

Proof of Theorem 1. As in the proof of Proposition 2 we set $\mathcal{F}_T = C([0,T];H_0^2(0,L)) \cap L^2(0,T;H^4(0,L))$. The hypotheses of that proposition are fulfilled, and hence, Eq. (1) with $\gamma = \gamma(t,x)$ has a unique solution $z \in \mathcal{F}_T$. Therefore, it only left to prove that $z_t \in \mathcal{F}_T$ and $z \in C([0,T];H^6(0,L))$.

Let $y := z_t - \psi_t$. From Eq. (1) it follows that y satisfies the equation

$$\begin{cases} y_t + (\sigma y_{xx})_{xx} + \gamma y_{xx} + z y_x + z_x y = g, & (t,x) \in (0,T) \times (0,L), \\ y(t,0) = 0, \quad y(t,L) = 0, & t \in (0,T), \\ y_x(t,0) = 0, \quad y_x(t,L) = 0, & t \in (0,T), \\ y(x,0) = y_0(x), & x \in (0,L), \end{cases} \tag{31}$$

where $g := f_t - \gamma_t z_{xx} - \psi_{tt} - (\sigma \psi_{txx})_{xx} - \gamma \psi_{txx} - z\psi_{tx} - z_x \psi_t$ and $y_0(x) := z_t(0,x) - \psi_t(0,x)$. Note that from Eq. (1) we have $y_0(x) = f(0,x) - \left(\sigma(x) z_0''(x)\right)'' - \gamma(0,x) z_0''(x) - z_0(x) z_0'(x) - \psi_t(0,x)$. By using the continuous injection $H^1(0,L) \hookrightarrow L^\infty(0,L)$ we obtain

$$\begin{aligned} \|g\|_{L^2(0,T;L^2(0,L))} \leq C \Big(& \|f_t\|_{L^2(0,T;L^2(0,L))} + \|\gamma\|_{H^1(0,T;H^1(0,L))} \|z\|_{C([0,T];H^2(0,L))} + \|(h_1,h_2,h_3,h_4)\|_{H^2(0,T)^4} \\ & + \|z\|_{C([0,T];H^1(0,L))} \|(h_1,h_2,h_3,h_4)\|_{H^1(0,T)^4} \Big), \end{aligned} \tag{32}$$

$$\begin{aligned} \|y_0\|_{H^2(0,L)} \leq C \Big(& \|f\|_{C([0,T];H^2(0,L))} + \|\sigma\|_{H^4(0,L)} \|z_0\|_{H^6(0,L)} + \|\gamma\|_{C([0,T];H^2(0,L))} \|z_0\|_{H^4(0,L)} \\ & + \|z_0\|^2_{H^3(0,L)} + \|(h_1,h_2,h_3,h_4)\|_{W^{1,\infty}(0,T)^4} \Big). \end{aligned} \tag{33}$$

These inequalities allow us to conclude that $g \in L^2(0,T;L^2(0,L))$. Moreover, $y_0 \in H_0^2(0,L)$ is obtained by additionally considering the compatibility condition. Therefore, we can use Lemma 1, with $\beta = z$ and $\alpha = 0$, on Eq. (31) to obtain

$$\|y\|_{\mathcal{Y}_T} \leq \exp\left\{C(1+T)\left(1 + \|z\|_{L^\infty(0,T;H^2(0,L))}\right)\right\} \left(\|g\|_{L^2(0,T;L^2(0,L))} + \|y_0\|_{H^2(0,L)}\right).$$

Plugging (22), (32) and (33) on this equation, then considering $\|z_t\|_{\mathcal{Y}_T} - \|\psi_t\|_{\mathcal{Y}_T} \leq \|y\|_{\mathcal{Y}_T}$ and finally using the inequality $x \leq e^x$ for all $x \in \mathbb{R}$ give us

$$\|z_t\|_{\mathcal{Y}_T} \leq \exp\left\{\exp\left\{C\left(1 + \|f\|_{C([0,T];H^2(0,L)) \cap H^1(0,T;L^2(0,L))} + \|(h_1,h_2,h_3,h_4)\|_{H^2(0,T)^4} + \|z_0\|_{H^6(0,L)}\right)\right\}\right\}. \tag{34}$$

Now it only rest to prove that $z \in C([0,T];H^6(0,L))$. For doing this we first see that $z \in C([0,T];H^4(0,L))$ because of the continuous injection $H^1(0,T;H^4(0,L)) \hookrightarrow C([0,T];H^4(0,L))$. From Eq. (1) we have

$$\begin{aligned} \|(\sigma z_{xx})_{xx}\|_{C([0,T];H^2(0,L))} \leq \ & \|f\|_{C([0,T];H^2(0,L))} + \|z_t\|_{C([0,T];H^2(0,L))} \\ & + \|\gamma\|_{L^\infty(0,T;L^\infty(0,L))} \|z\|_{C([0,T];H^4(0,L))} + \|z\|^2_{C([0,T];H^3(0,L))}. \end{aligned}$$

Therefore, plugging (34) on this inequality and then using the inequality $x \leq e^x$ for all $x \in \mathbb{R}$ give us

$$\begin{aligned} \|(\sigma z_{xx})_{xx}\|_{C([0,T];H^2(0,L))} \leq \exp\Big\{\exp\Big\{C\Big(& 1 + \|f\|_{C([0,T];H^2(0,L)) \cap H^1(0,T;L^2(0,L))} \\ & + \|(h_1,h_2,h_3,h_3)\|_{H^2(0,T)^4} + \|z_0\|_{H^6(0,L)}\Big)\Big\}\Big\}. \end{aligned} \tag{35}$$

We can use the same argument based on Ehrling's lemma (see Part 2 of the proof of Proposition 1) to conclude that $z \in C([0,T];H^6(0,L))$. Finally, (5) is a consequence of (34) and (35) after the above-mentioned argument. The proof of Theorem 1 is complete. □

3. Carleman estimates

In this section we provide the suitable Carleman estimates for the study of the stability of our inverse problem. We begin with an estimate for fourth order parabolic operators and finish with an estimate for a special case of second order elliptic operators. Both estimates are obtained by following a procedure due to A.V. Fursikov and O.Y. Imanuvilov [9].

For $s > 0$ we define the weight functions

$$\phi(t,x) := \frac{e^{2s\|\beta\|_{L^\infty(0,L)}} - e^{s\beta(x)}}{\alpha(t)}, \qquad \eta(t,x) := \frac{e^{s\beta(x)}}{\alpha(t)},$$

where $\alpha \in C^1([0,T])$ and $\beta \in C^2([0,L])$ are such that

- $\alpha(0) = \alpha(T) = 0$, and for $T_0 \in (0,T)$ given, $0 < \alpha(t) \leq \alpha(T_0)$ for every $t \in (0,T)$.
- For $r > 0$ given, $r \leq \beta(x)$, $\beta_x(x) \leq -r$ and $\beta_{xx}(x) = 0$ for every $x \in (0,L)$.

From these conditions it follows that $\phi > 0$ and $\eta > 0$ on $(0,T) \times (0,L)$. Also $\phi(t,x) \geq \phi(T_0,x)$ for every $(t,x) \in (0,T) \times (0,L)$ is obtained. Note that ϕ and η are unbounded in $(0,T) \times (0,L)$ because both blows up at $t = 0$ and $t = T$, however, $e^{-\lambda\phi}\eta^k$ is bounded for every $k \in \mathbb{N} \cup \{0\}$ and $\lambda > 0$. Concerning the latter we have

$$\lim_{t \downarrow 0} e^{-\lambda\phi(t,x)} = \lim_{t \uparrow T} e^{-\lambda\phi(t,x)} = 0, \quad \forall x \in (0,L). \tag{36}$$

Some computations which are not difficult allow us to obtain the next lemma, which is presented without proof, where we list some useful properties of the weight functions that will be used from now on.

Lemma 2. *In* $(0,T)\times(0,L)$ *the following holds:*

(a) $s \le \left(\|\alpha\|_{L^\infty(0,T)}/r\right)\eta$.
(b) $\partial_x^k\phi = -(\beta_x)^k s^k\eta$ *for every* $k\in\mathbb{N}$.
(c) $|\phi_{xt}| \le \left(\|\alpha_t\|_{L^\infty(0,T)}\|\beta_x\|_{L^\infty(0,L)}/r\right)\eta^2$.
(d) $|\phi_t| \le \left(\|\alpha_t\|_{L^\infty(0,T)}/r^2\right)\lambda s^{-2}\eta^2$ *if* $\lambda \ge e^{2s\|\beta\|_{L^\infty(0,L)}}$.

In what follows we set $Q_T := (0,T)\times(0,L)$. Also, the letter C will denote a positive constant, independent of s and λ, which may vary from line to line. Finally, we assume that α and β are fixed.

3.1. Fourth order parabolic operators

Let $(q_1,q_2,q_3)\in L^\infty(0,T;L^\infty(0,L))^3$, $\sigma_0>0$ and $\sigma\in W^{1,\infty}(0,T;W^{3,\infty}(0,L))$ be such that $\sigma(t,x)\ge\sigma_0$ for every $(t,x)\in Q_T$. Consider the operator

$$Pv := v_t + \sigma v_{xxxx} + q_3 v_{xxx} + q_2 v_{xx} + q_1 v_x + q_0 v,$$

defined on $\mathcal{V} := \{v\in L^2(0,T;H^4\cap H_0^2(0,L))/Pv\in L^2(0,T;L^2(0,L))\}$. Note that if $v\in\mathcal{V}$, then $v_t\in L^2(0,T;L^2(0,L))$. In the next proposition we present a Carleman estimate with two large dependent parameters.

Proposition 3. *Let* $m>0$, $\|(q_1,q_2,q_3)\|_{L^\infty(0,T;L^\infty(0,L))^3}\le m$ *and* $\|\sigma\|_{W^{1,\infty}(0,T;W^{3,\infty}(0,L))}\le m$. *There exist* $s_0>0$ *and* $C=C(m,\sigma_0,s_0)>0$ *such that for every* $s\ge s_0$, $\lambda\ge e^{2s\|\beta\|_{L^\infty(0,L)}}$ *and* $v\in\mathcal{V}$ *we have*

$$\iint_{Q_T} e^{-2\lambda\phi}\left(\frac{|v_t|^2+|v_{xxxx}|^2}{\lambda\eta}\right)dxdt + \iint_{Q_T} e^{-2\lambda\phi}\left(\lambda^7 s^8\eta^7|v|^2+\lambda^5 s^6\eta^5|v_x|^2+\lambda^3 s^4\eta^3|v_{xx}|^2+\lambda s^2\eta|v_{xxx}|^2\right)dxdt$$
$$\le C\iint_{Q_T} e^{-2\lambda\phi}|Pv|^2dxdt + C\int_0^T e^{-2\lambda\phi(t,0)}\left(\lambda^3 s^3\eta^3|v_{xx}(t,0)|^2+\lambda s\eta|v_{xxx}(t,0)|^2\right)dt. \tag{37}$$

Proof. It is enough to prove (37) for $P_p v := v_t + \sigma v_{xxxx}$ with $v\in\mathcal{V}$. In fact, assume that we have proved (37) for $P_p v$ instead of Pv. We have

$$\iint_{Q_T} e^{-2\phi\lambda}|P_p v|^2 \le 2\iint_{Q_T}|Pv|^2dxdt + C\iint_{Q_T} e^{-2\phi\lambda}(|v_{xxx}|^2+|v_{xx}|^2+|v_x|^2+|v|^2)dxdt.$$

By choosing $s>0$ and $\lambda>0$ large, and taking into account Lemma 2, it is possible to absorb

$$C\iint_{Q_T} e^{-2\phi\lambda}(|v_{xxx}|^2+|v_{xx}|^2+|v_x|^2+|v|^2)dxdt,$$

with the left-hand side of (37), concluding that (37) also holds for Pv.

By following a procedure due to A.V. Fursikov and O.Y. Imanuvilov [9], we will prove a Carleman estimate for the operator $Lw := e^{-\lambda\phi}P_p(e^{\lambda\phi}w)$ defined on $\mathcal{W} := \{e^{-\lambda\phi}v/v\in\mathcal{V}\}$. This will lead us naturally to (37) for $P_p v$.

Consider the decomposition $Lw = L_1w + L_2w + R_0w$, where

$$\begin{aligned}
L_1w &:= w_t + 4\lambda^3\phi_x^3 w_x + 4\lambda\phi_x w_{xxx} + 4\lambda^3\phi_x(\phi_x^2\sigma)_x w,\\
L_2w &:= \lambda^4\phi_x^4 w + 6\lambda^2\phi_x^2 w_{xx} + \sigma w_{xxxx} + 6\lambda^2(\phi_x^2\sigma)_x w_x,\\
R_0w &:= \lambda\phi_t w + \lambda\phi_{xxxx}\sigma w + 4\lambda^2\phi_{xxx}\phi_x\sigma w + 4\lambda\phi_{xxx}\sigma w_x + 6\lambda^3\phi_{xx}\phi_x^2\sigma w + 12\lambda^2\phi_{xx}\phi_x\sigma w_x\\
&\quad + 6\lambda\phi_{xx}\sigma w_{xx} + 3\lambda^2\phi_{xx}^2\sigma w - 4\lambda^3\phi_x(\phi_x^2\sigma)_x w - 6\lambda^2(\phi_x^2\sigma)_x w_x.
\end{aligned}$$

The key for obtaining the Carleman estimate is to correctly choose L_1w, L_2w and R_0w. The structure of the decomposition gives us

$$\|Lw - R_0w\|_{L^2(Q_T)}^2 = \|L_1w\|_{L^2(Q_T)}^2 + 2\,(L_1w, L_2w)_{L^2(Q_T)} + \|L_2w\|_{L^2(Q_T)}^2. \tag{38}$$

Now we introduce some notations that will be used throughout this proof. The purpose of doing this is to make clear all the computations needed. For $w\in\mathcal{W}$ we introduce

$$\|w\|_A := \iint_{Q_T}\left(\lambda^7 s^8\eta^7|w|^2+\lambda^5 s^6\eta^5|w_x|^2+\lambda^3 s^4\eta^3|w_{xx}|^2+\lambda s^2\eta|w_{xxx}|^2\right)dxdt,$$
$$\|w\|_B := \iint_{Q_T}\left(\frac{|w_t|^2+|w_{xxxx}|^2}{\lambda\eta}\right)dxdt,$$

$$I_0(w) := -\lambda^7 \iint_{Q_T} 6\phi_x^6\phi_{xx}\sigma^2|w|^2dxdt, \tag{39}$$

$$I_1(w) := -\lambda^5 \iint_{Q_T} \left(30\phi_x^4\phi_{xx}\sigma^2 + 12\phi_x^5\sigma\sigma_x\right)|w_x|^2dxdt, \tag{40}$$

$$I_2(w) := -\lambda^3 \iint_{Q_T} \left(58\phi_x^2\phi_{xx}\sigma^2 + 32\phi_x^3\sigma\sigma_x\right)|w_{xx}|^2dxdt, \tag{41}$$

$$I_3(w) := -\lambda \iint_{Q_T} \left(2\phi_{xx}\sigma^2 + 4\phi_x\sigma\sigma_x\right)|w_{xxx}|^2dxdt, \tag{42}$$

and finally

$$B(w,x) := \int_0^T \left(10\lambda^3\phi_x^3(t,x)\sigma^2(t,x)|w_{xx}(t,x)|^2 + 2\lambda\phi_x(t,x)\sigma^2(t,x)|w_{xxx}(t,x)|^2\right)dt.$$

Before continuing with this proof we need the following two technical lemmas.

Lemma 3. *Under the hypotheses of Proposition 3, there exist $s_0 > 0$ and $C = C(m, \sigma_0, s_0) > 0$ such that for every $s \geq s_0$, $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$ and $w \in \mathcal{W}$ we have*

(a) $\|R_0 w\|^2_{L^2(Q_T)} \leq C\lambda^{-1}s^{-1}\|w\|_A$.

(b) $\|w\|_B \leq C\left(\|L_1 w\|^2_{L^2(Q_T)} + \|L_2 w\|^2_{L^2(Q_T)} + \|w\|_A\right)$.

(c) $I_3(w) + I_2(w) + I_1(w) + I_0(w) \geq C\|w\|_A$.

Proof. First, part (a) follows from a direct application of Lemma 2 by considering $s > 0$ large and $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$. Note that part (d) of Lemma 2 has been only used to bound the term $\lambda\phi_t w$. Second, if we consider $s > 0$ and $\lambda > 0$ large with part (b) of Lemma 2 we get

- $\frac{1}{C}\iint_{Q_T} \frac{|w_t|^2}{\lambda\eta}dxdt \leq \iint_{Q_T}|L_1 w|^2dxdt + \iint_{Q_T}\left(\lambda^5 s^6\eta^5|w_x|^2 + \lambda s^2\eta|w_{xxx}|^2 + \lambda^5 s^6\eta^5|w|^2\right)dxdt.$
- $\frac{1}{C}\iint_{Q_T} \frac{|w_{xxxx}|^2}{\lambda\eta}dxdt \leq \iint_{Q_T}|L_2 w|^2dxdt + \iint_{Q_T}\left(\lambda^7 s^8\eta^7|w|^2 + \lambda^3 s^4\eta^3|w_{xx}|^2 + \lambda^3 s^4\eta^3|w_x|^2\right)dxdt.$

These two inequalities give us part (b). Finally, from (39) to (42) and part (b) of Lemma 2 we conclude that there exist $C_1 = C_1(\sigma_0) > 0$ and $C_2 = C_2(m) > 0$ such that

$$I_3(w) + I_2(w) + I_1(w) + I_0(w) \geq C_1\|w\|_A - C_2 s^{-1}\|w\|_A.$$

Therefore, by choosing $s_0 > 0$ large we obtain the existence of $C = C(\sigma_0, m, s_0) > 0$ such that for every $s \geq s_0$ we have part (c). The proof of Lemma 3 is complete. □

Lemma 4. *Under the hypotheses of Proposition 3, for every $w \in \mathcal{W}$ we have*

$$(L_1 w, L_2 w)_{L^2(Q_T)} = \sum_{k=0}^{3} I_k(w) + B(w,L) - B(w,0) + R_1(w), \tag{43}$$

where $R_1(w)$, which is defined in (45), gathers terms that for $s > 0$ and $\lambda > 0$ large satisfies

$$|R_1(w)| \leq Cs^{-1}\left(\|w\|_A + \|w\|_B\right). \tag{44}$$

Proof. For $i, j = 1, 2, 3, 4$ we denote by $I_{i,j}$ the L^2-product in Q_T between the i-th term of $L_1 w$ with the j-th term of $L_2 w$. Note that with this we have

$$(L_1 w, L_2 w)_{L^2(Q_T)} = \sum_{i,j=1}^{4} I_{i,j}.$$

Integrations by parts are performed in time or space on every $I_{i,j}$ and each resulting term is included in only one part of the right-hand side of (43). Since $v \in \mathcal{V}$ implies $v_t \in L^2(0,T;L^2(0,L))$, we conclude that $v \in C([0,T];L^2(0,L))$. Therefore, in virtue of (36), every $w \in \mathcal{W}$ satisfies $w \in C([0,T];L^2(0,L))$ with $w(0,x) = w(T,x) = 0$ a.e. $x \in (0,L)$. Each resulting expression for $I_{i,j}$ is listed below.

$$\bullet\ I_{1,1} = \underbrace{-\frac{\lambda^4}{2}\iint_{Q_T}\left(\phi_x^4\sigma\right)_t|w|^2dxdt}_{R_1(w)}.$$

- $I_{1,2} = -I_{1,4} + \underbrace{3\lambda^2 \iint_{Q_T} \left(\phi_x^2\sigma\right)_t |w_x|^2 dxdt}_{R_1(w)}.$

- $I_{1,3} = \underbrace{-\iint_{Q_T} \sigma_{xx} w_{xx} w_t \, dxdt}_{R_1(w)} \underbrace{-2\iint_{Q_T} \sigma_x w_{xxx} w_t \, dxdt}_{R_1(w)} \underbrace{-\frac{1}{2}\iint_{Q_T} \sigma_t |w_{xx}|^2 dxdt}_{R_1(w)}.$

- $I_{1,4} = 6\lambda^2 \iint_{Q_T} \left(\phi_x^2\sigma\right)_x w_x w_t \, dxdt,$ which is canceled when adding $I_{1,2}$.

- $I_{2,1} = \underbrace{-2\lambda^7 \iint_{Q_T} \left(\phi_x^7\sigma^2\right)_x |w|^2 dxdt}_{I_0(w)}.$

- $I_{2,2} = \underbrace{-12\lambda^5 \iint_{Q_T} \left(\phi_x^5\sigma^2\right)_x |w_x|^2 dxdt}_{I_1(w)}.$

- $$I_{2,3} = \underbrace{6\lambda^3 \iint_{Q_T} \left(\phi_x^3\sigma^2\right)_x |w_{xx}|^2 dxdt}_{I_2(w)} \underbrace{-2\lambda^3 \iint_{Q_T} \left(\phi_x^3\sigma^2\right)_{xxx} |w_x|^2 dxdt}_{R_1(w)}$$
$$\underbrace{-2\lambda^3 \int_0^T \phi_x^3(t,L)\sigma^2(t,L)|w_{xx}(t,L)|^2 dt}_{B(w,L)} + \underbrace{2\lambda^3 \int_0^T \phi_x^3(t,0)\sigma^2(t,0)|w_{xx}(t,0)|^2 dt}_{B(w,0)}.$$

- $I_{2,4} = 24\lambda^5 \iint_{Q_T} \left(\phi_x^2\sigma\right)_x \phi_x^3\sigma |w_x|^2 dxdt,$ which is canceled when adding $I_{4,2}$.

- $I_{3,1} = \underbrace{-2\lambda^5 \iint_{Q_T} \left(\phi_x^5\sigma^2\right)_{xxx} |w|^2 dxdt}_{R_1(w)} + \underbrace{6\lambda^5 \iint_{Q_T} \left(\phi_x^5\sigma^2\right)_x |w_x|^2 dxdt}_{I_1(w)}.$

- $$I_{3,2} = \underbrace{-12\lambda^3 \iint_{Q_T} \left(\phi_x^3\sigma^2\right)_x |w_{xx}|^2 dxdt}_{I_2(w)} + \underbrace{12\lambda^3 \int_0^T \phi_x^3(t,L)\sigma^2(t,L)|w_{xx}(t,L)|^2 dt}_{B(w,L)}$$
$$\underbrace{-12\lambda^3 \int_0^T \phi_x^3(t,0)\sigma^2(t,0)|w_{xx}(t,0)|^2 dt}_{B(w,0)}.$$

- $$I_{3,3} = \underbrace{-2\lambda \iint_{Q_T} \left(\phi_x\sigma^2\right)_x |w_{xxx}|^2 dxdt}_{I_3(w)} + \underbrace{2\lambda \int_0^T \phi_x(t,L)\sigma^2(t,L)|w_{xxx}(t,L)|^2 dt}_{B(w,L)}$$
$$\underbrace{-2\lambda \int_0^T \phi_x(t,0)\sigma^2(t,0)|w_{xxx}(t,0)|^2 dt}_{B(w,0)}.$$

- $I_{3,4} = \underbrace{12\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} |w_x|^2 dxdt}_{R_1(w)} \underbrace{-24\lambda^3 \iint_{Q_T} \left(\phi_x^2\sigma\right)_x \phi_x\sigma |w_{xx}|^2 dxdt}_{I_2(w)}.$

- $I_{4,1} = \underbrace{4\lambda^7 \iint_{Q_T} \phi_x^5\sigma \left(\phi_x^2\sigma\right)_x |w|^2 dxdt}_{I_0(w)}.$

- $I_{4,2} = \underbrace{12\lambda^5 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x^3\sigma\right]_{xx} |w|^2 dxdt}_{R_1(w)} - I_{2,4}.$

$$\bullet\, I_{4,3} = \underbrace{4\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} w w_{xx}\, dxdt}_{R_1(w)} \underbrace{-4\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} |w_x|^2 dxdt}_{R_1(w)}$$

$$+ \underbrace{4\lambda^3 \iint_{Q_T} \left(\phi_x^2\sigma\right)_x \phi_x\sigma |w_{xx}|^2 dxdt}_{I_2(w)}.$$

$$\bullet\, I_{4,4} = \underbrace{-12\lambda^5 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x^3\sigma\right]_x |w|^2 dxdt}_{R_1(w)}.$$

By adding all the above terms we obtain (43) and conclude that

$$\begin{aligned} R_1(w) := &-\frac{\lambda^4}{2} \iint_{Q_T} \left(\phi_x^4\sigma\right)_t |w|^2 dxdt - 2\lambda^5 \iint_{Q_T} \left(\phi_x^5\sigma^2\right)_{xxx} |w|^2 dxdt + 12\lambda^5 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x^3\sigma\right]_{xx} |w|^2 dxdt \\ &- 12\lambda^5 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x^3\sigma\right]_x |w|^2 dxdt + 3\lambda^2 \iint_{Q_T} \left(\phi_x^2\sigma\right)_t |w_x|^2 dxdt - 2\lambda^3 \iint_{Q_T} \left(\phi_x^3\sigma^2\right)_{xxx} |w_x|^2 dxdt \\ &+ 12\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} |w_x|^2 dxdt - 4\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} |w_x|^2 dxdt - \frac{1}{2} \iint_{Q_T} \sigma_t |w_{xx}|^2 dxdt \\ &- \iint_{Q_T} \sigma_{xx} w_{xx} w_t\, dxdt - 2 \iint_{Q_T} \sigma_x w_{xxx} w_t\, dxdt + 4\lambda^3 \iint_{Q_T} \left[\left(\phi_x^2\sigma\right)_x \phi_x\sigma\right]_{xx} w w_{xx}\, dxdt. \end{aligned} \quad (45)$$

Consider $s > 0$ and $\lambda > 0$ large. According to Lemma 2, all except the last three terms of the right-hand side of (45) can be bounded by $C\lambda^{-2}s^{-2}\|w\|_A$. The three remaining terms can be bounded by $Cs^{-1}\left(\|w\|_A + \|w\|_B\right)$ by additionally using the Cauchy–Schwarz inequality. This allows us to get (44). The proof of Lemma 4 is complete. □

Now we continue with the proof of Proposition 3. In what follows we consider $s > 0$ large and $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$. Since for every $w \in \mathcal{W}$ and $x \in (0, L)$ we have $B(w, x) \geq 0$, we can use (38), (43) of Lemma 4 and part (c) of Lemma 3 to obtain

$$C\left(\|L_1 w\|^2_{L^2(Q_T)} + \|L_2 w\|^2_{L^2(Q_T)} + \|w\|_A\right) \leq 2\|Lw\|^2_{L^2(Q_T)} + 2\|R_0 w\|^2_{L^2(Q_T)} + B(w, 0) + |R_1(w)|.$$

Combining this inequality with part (b) of Lemma 3, part (a) of Lemma 3 and (44) of Lemma 4 we get

$$\|w\|_A + \|w\|_b \leq C\left(\|Lw\|^2_{L^2(Q_T)} + B(w, 0)\right) + C\left(\lambda^{-1}s^{-1} + s^{-1}\right)\left(\|w\|_A + \|w\|_b\right). \quad (46)$$

Since

$$|B(w, x)| \leq C \int_0^T \left(\lambda^3 s^3 \eta^3(t, x)|w_{xx}(t, x)|^2 + \lambda s\eta(t, x)|w_{xxx}(t, x)|^2\right) dt,$$

we see from (46) that we can choose $s_0 > 0$ large to obtain that there exists $C = C(m, \sigma_0, s_0) > 0$ such that for every $s \geq s_0$, $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$ and $w \in \mathcal{W}$ we have

$$\|w\|_A + \|w\|_b \leq C\left(\|Lw\|^2_{L^2(Q_T)} + \int_0^T \left(\lambda^3 s^3 \eta^3(t, 0)|w_{xx}(t, 0)|^2 + \lambda s\eta(t, 0)|w_{xxx}(t, 0)|^2\right) dt\right). \quad (47)$$

This is the Carleman estimate for Lw. Now we have to return to the original variable v and retrieve from (47) the Carleman estimate for $P_p v$. By taking into account Lemma 2, the following inequalities are obtained:

- $\iint_{Q_T} e^{-2\lambda\phi} \frac{1}{\lambda\eta}\left(|(e^{\lambda\phi}w)_t|^2 + |(e^{\lambda\phi}w)_{xxxx}|^2\right) dxdt \leq C\left(\|w\|_A + \|w\|_B\right)$.
- $\iint_{Q_T} e^{-2\lambda\phi}\left(\lambda^7 s^8 \eta^7 |e^{\lambda\phi}w| + \lambda^5 s^6 \eta^5 |(e^{\lambda\phi}w)_x| + \lambda^3 s^4 \eta^3 |(e^{\lambda\phi}w)_{xx}| + \lambda s^2 \eta |(e^{\lambda\phi}w)_{xxx}|\right) dxdt \leq C\|w\|_A$.
- $\int_0^T \lambda^3 s^3 \eta^3(t, 0)|w_{xx}(t, 0)|^2 dt = \int_0^T e^{-2\lambda\phi(t,0)} \lambda^3 s^3 \eta^3(t, 0)|v_{xx}(t, 0)|^2 dt$.
- $\int_0^T \lambda s\eta(t, 0)|w_{xxx}(t, 0)|^2 dt \leq C \int_0^T e^{-2\lambda\phi(t,0)}\left(\lambda^3 s^3 \eta^3(t, 0)|v_{xx}(t, 0)|^2 + \lambda s\eta(t, 0)|v_{xxx}(t, 0)|^2\right) dt$.

The Carleman estimate for $P_p v$ is obtained by combining these inequalities with (47) and by considering $w = e^{-\lambda\phi}v$ and $Lw := e^{-\lambda\phi}P_p(e^{\lambda\phi}w)$. The proof of Proposition 3 is complete. □

3.2. Second order elliptic operators

Let $R_0 > 0$ and $R \in H^2(0, L)$ be such that $|R(x)| \geq R_0$ for every $x \in (0, L)$. Consider the operator

$$Ef := (Rf)_{xx},$$

defined on $\mathcal{F} := \{f \in H^1(0, L)/Ef \in L^2(0, L)\}$. Note that if $f \in \mathcal{F}$, then $f_{xx} \in L^2(0, L)$. In this section we consider $\varphi(x) := \phi(T_0, x)$ and $\nu(x) := \eta(T_0, x)$ as weight functions. For these weight functions, Lemma 2 is still valid. In the next proposition we present a Carleman estimate with two large independent parameters.

Proposition 4. *Let $m > 0$ and $\|R\|_{H^2(0,L)} \leq m$. There exist $s_0 > 0$, $\lambda_0 > 0$ and $C = C(m, R_0, s_0, \lambda_0) > 0$ such that for every $s \geq s_0$, $\lambda \geq \lambda_0$ and $f \in \mathcal{F}$ we have*

$$\int_0^L e^{-2\lambda\varphi} \left(\lambda^3 s^4 \nu^3 |f|^2 + \lambda s^2 \nu |f_x|^2\right) dx \leq C \int_0^L e^{-2\lambda\varphi} |Ef|^2 dx + Ce^{-2\lambda\varphi(0)} \left(\lambda^3 s^3 \nu^3(0) |f(0)|^2 + \lambda s \nu(0) |f_x(0)|^2\right). \quad (48)$$

Remark 4. Note that R, the main coefficient of the operator E, belongs to $H^2(0, L)$ and not to $W^{2,\infty}(0, L)$ in contrast with σ, the main coefficient of operator the P, which is in $W^{1,\infty}(0, T; W^{3,\infty}(0, L))$. This regularity for R will be essential in the study of the stability of our inverse problem.

Proof of Proposition 4. We proceed as we did in the proof of Proposition 3. Let $E_p f := Rf_{xx}$ with $f \in \mathcal{F}$. Assume that we have proved (48) for $E_p f$ instead of Ef. Since $Ef = E_p f + 2R_x f_x + R_{xx} f$, we have

$$\int_0^L e^{-2\lambda\varphi} |E_p f|^2 dx \leq 2 \int_0^L e^{-2\lambda\varphi} |Ef|^2 dx + 8\|R_x\|^2_{L^\infty(0,L)} \int_0^L e^{-2\lambda\varphi} |f_x|^2 dx + 2\|R_{xx}\|^2_{L^2(0,L)} \|e^{-\lambda\varphi} f\|^2_{L^\infty(0,L)}. \quad (49)$$

Considering the continuous injection $H^1(0, L) \hookrightarrow L^\infty(0, L)$, the last two terms of the right-hand side of (49) can be bounded from above by

$$C \int_0^L e^{-2\lambda\varphi} \left(|f|^2 + \lambda^2 s^2 \nu^2 |f|^2 + |f_x|^2\right) dx. \quad (50)$$

By choosing $s > 0$ and $\lambda > 0$ large, and taking into account Lemma 2, (50) can be absorbed with the left-hand side of (48), concluding that (48) also holds for Ef. Therefore, we only have to prove (48) for $E_p f$. As we did before, we will prove a Carleman estimate for the operator $Lw := e^{-\lambda\varphi} E_p(e^{\lambda\varphi} w)/R$ defined on $\mathcal{W} := \{e^{-\lambda\phi} f / f \in \mathcal{F}\}$. This will lead us naturally to (48) for $E_p v$.

Consider the decomposition $Lw = L_1 w + L_2 w + R_0 w$, where

$$L_1 w := \lambda^2 \varphi_x^2 w + w_{xx},$$
$$L_2 w := 2\lambda \varphi_x w_x,$$
$$R_0 w := \lambda \varphi_{xx} w.$$

The structure of the decomposition gives us

$$\|Lw - R_0 w\|^2_{L^2(0,L)} = \|L_1 w\|^2_{L^2(0,L)} + 2\,(L_1 w, L_2 w)_{L^2(0,L)} + \|L_2 w\|^2_{L^2(0,L)}. \quad (51)$$

We have that

$$\begin{aligned}(L_1 w, L_2 w)_{L^2(0,L)} = & -3 \int_0^L \lambda^3 \varphi_x^2 \varphi_{xx} |w|^2 dx + \lambda^3 \varphi_x^3(L) |w(L)|^2 - \lambda^3 \varphi_x^3(0) |w(0)|^2 \\ & - \int_0^L \lambda \varphi_{xx} |w_x|^2 dx + \lambda \varphi_x(L) |w_x(L)|^2 - \lambda \varphi_x(0) |w_x(0)|^2.\end{aligned} \quad (52)$$

If for $w \in \mathcal{W}$ we define

$$\|w\|_A := \int_0^L \left(\lambda^3 s^4 \eta^3 |w|^2 + \lambda s^2 \eta |w_x|^2\right) dx,$$

then from (51) and (52) we obtain, by taking into account Lemma 2, that

$$\|w\|_A \leq C \left(\|Lw\|^2_{L^2(0,L)} + \|R_0 w\|^2_{L^2(0,L)} + \lambda^3 s^3 \nu^3(0) |w(0)|^2 + \lambda s \nu(0) |w_x(0)|^2\right).$$

Combining this inequality with

$$\|R_0 w\|^2_{L^2(0,L)} \leq C \lambda^{-1} s^{-2} \|w\|_A,$$

we see that we can choose $s_0 > 0$ and $\lambda_0 > 0$ large to obtain that there exists $C = C(m, R_0, s_0, \lambda_0) > 0$ such that for every $s \geq s_0$, $\lambda \geq \lambda_0$ and $w \in \mathcal{W}$ we have

$$\|w\|_A \leq C\left(\|Pw\|^2_{L^2(0,L)} + \lambda^3 s^3 \nu^3(0)|w(0)|^2 + \lambda s\nu(0)|w_x(0)|^2\right).$$

To return to the original variable f, which will allow us to get the Carleman estimate for $E_p f$, we proceed in the same way as we did at the end of the proof of Proposition 3. The proof of Proposition 4 is complete. □

The following result, which is presented without proof, is a direct consequence of Proposition 4 by taking into account the hypothesis $|R(x)| \geq R_0 > 0$ for every $x \in (0, L)$. This hypothesis allows us to include the term $e^{-2\lambda\varphi}|f_{xx}|^2$ into the left-hand side of (48) since

$$R_0^2 \int_0^L e^{-2\lambda\varphi}|f_{xx}|^2 dx \leq \int_0^L e^{-2\lambda\varphi}|E_p f|^2 dx.$$

(See the notation introduced at the beginning of the proof of Proposition 4.)

Proposition 5. *Let $m > 0$ and $\|R\|_{H^2(0,L)} \leq m$. There exist $s_0 > 0$, $\lambda_0 > 0$ and $C = C(m, R_0, s_0, \lambda_0) > 0$ such that for every $s \geq s_0$, $\lambda \geq \lambda_0$ and $f \in \mathcal{F}$ we have*

$$\int_0^L e^{-2\lambda\varphi}\left(\lambda^3 s^4 \nu^3|f|^2 + \lambda s^2\nu|f_x|^2 + |f_{xx}|^2\right) dx \leq C\int_0^L e^{-2\lambda\varphi}|Ef|^2 dx + Ce^{-2\lambda\varphi(0)} \times \left(\lambda^3 s^3 \nu^3(0)|f(0)|^2 + \lambda s\nu(0)|f_x(0)|^2\right). \tag{53}$$

Remark 5. If we consider β to be an increasing function, then inequalities (37), (48) and (53) with boundary terms at $x = L$ instead of $x = 0$ would have been obtained. As can be seen in the proof of Theorem 2, the boundary terms in the Carleman estimates are related to the location of the measurements of the inverse problem.

4. Stability of the inverse problem

This section is devoted to the proof of Theorem 2, which is a local Lipschitz stability result in Σ_{ad} for our inverse problem. To prove this theorem we use a method due to Bukhgeim–Klibanov [4] and the Carleman estimates given by Propositions 3 and 5.

Proof of Theorem 2. The proof of this proposition is split into four parts.

Part 1: *Method of Bukhgeim–Klibanov.* Let $u := z(\sigma) - z(\overline{\sigma})$, $f := \overline{\sigma} - \sigma$ and $R := z_{xx}(\overline{\sigma})$. It follows from Eq. (1) that u satisfies the equation

$$\begin{cases} u_t + (\sigma u_{xx})_{xx} + \gamma u_{xx} + z(\sigma)u_x + z_x(\sigma)u = (fR)_{xx}, & (t,x) \in (0,T)\times(0,L),\\ u(t,0) = 0, \quad u(t,L) = 0, & t \in (0,T),\\ u_x(t,0) = 0, \quad u_x(t,L) = 0, & t \in (0,T),\\ u(0,x) = 0, & x \in (0,L). \end{cases} \tag{54}$$

Note that to prove this theorem it is sufficient to obtain an estimate of f in terms of $u_{xx}(\cdot, 0)$, $u_{xxx}(\cdot, 0)$ and $u(T_0, \cdot)$. Let $v := u_t$. It follows from Eq. (54) that v satisfies the equation

$$\begin{cases} v_t + (\sigma v_{xx})_{xx} + \gamma v_{xx} + z(\sigma)v_x + z_x(\sigma)v = (fR_t)_{xx} - h, & (t,x) \in (0,T)\times(0,L),\\ v(t,0) = 0, \quad v(t,L) = 0, & t \in (0,T),\\ v_x(t,0) = 0, \quad v_x(t,L) = 0, & t \in (0,T),\\ v(0,x) = (f(x)R(0,x))_{xx}, & x \in (0,L), \end{cases} \tag{55}$$

where $h := z_t(\sigma)u_x + z_{tx}(\sigma)u$.

Part 2: *Carleman estimate for elliptic operators.* Since $\sigma, \overline{\sigma} \in \Sigma_{ad}$, we see that $f \in H^4(0, L)$ with $f(0) = f_x(0) = 0$. This and the fact $R \in H^1(0, T; H^2(0, L))$ allow us to conclude that $(Ef)(x) := (f(x)R(T_0, x))_{xx} \in L^2(0, L)$. Also, by hypothesis we have $|R(T_0, x)| \geq R_0 > 0$ for every $x \in (0, L)$.

Therefore, if we define

$$N(f) := \int_0^L e^{-2\lambda\varphi}\left(\lambda^3 s^4\nu^3|f|^2 + \lambda s^2\nu|f_x|^2 + |f_{xx}|^2\right) dx,$$

then we can use the Carleman estimate given by Proposition 5 to obtain

$$N(f) \leq C\int_0^L e^{-2\lambda\phi(T_0,x)}|(f(x)R(T_0,x))_{xx}|^2 dx, \tag{56}$$

for $s > 0$ and $\lambda > 0$ large. We see that the boundary terms of (53) vanish because $f(0) = f_x(0) = 0$. This is the reason why we have asked prescribed values at $x = 0$ for the admissible main coefficients. By setting $C_1 := \|e^{-2\lambda\phi(T_0,\cdot)}\|_{L^\infty(0,L)}$, Eq. (54) at $t = T_0$ and (56) lead us to

$$\frac{1}{C}N(f) \leq C_1\|u(T_0,\cdot)\|^2_{H^4(0,L)} + \int_0^L e^{-2\lambda\phi(T_0,x)}|v(T_0,x)|^2 dx, \tag{57}$$

for $s > 0$ and $\lambda > 0$ large. Since $e^{-2\lambda\phi(0,x)}|v(0,x)|^2 = 0$, we get

$$\begin{aligned}\int_0^L e^{-2\lambda\phi(T_0,x)}|v(T_0,x)|^2 dx &= \iint_{Q_{T_0}} \frac{\partial}{\partial t}\left(e^{-2\lambda\phi}|v|^2\right) dtdx,\\ &= \iint_{Q_{T_0}} e^{-2\lambda\phi}\left(-2\lambda\phi_t\right)|v|^2 dxdt + 2\iint_{Q_{T_0}} e^{-2\lambda\phi} v v_t\, dxdt.\end{aligned} \tag{58}$$

Recall that $Q_T = (0,T) \times (0,L)$. In what follows we consider $s > 0$ large, $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$ and the properties of the weight functions given by Lemma 2. The following inequalities are obtained:

- $$\begin{aligned}\iint_{Q_{T_0}} e^{-2\lambda\phi}\left(-2\lambda\phi_t\right)|v|^2 dxdt &\leq C\iint_{Q_T} e^{-2\lambda\phi}\lambda^2 s^{-2}\eta^2|v|^2 dxdt,\\ &\leq C\lambda^{-5}s^{-15}\iint_{Q_T} e^{-2\lambda\phi}\lambda^7 s^8\eta^7|v|^2 dxdt.\end{aligned}$$
- $$\begin{aligned}\iint_{Q_{T_0}} e^{-2\lambda\phi} v v_t\, dxdt &\leq C\lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}\frac{1}{\lambda\eta}|v_t|^2 dxdt + C\lambda^3 s^7\iint_{Q_T} e^{-2\lambda\phi}\lambda\eta|v|^2 dxdt,\\ &\leq C\lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}\left(\frac{1}{\lambda\eta}|v_t|^2 + \lambda^7 s^8\eta^7|v|^2\right) dxdt.\end{aligned}$$

Therefore, plugging these inequalities in (58) and then the resulting terms in (57) give us

$$\frac{1}{C}N(f) \leq C_1\|u(T_0,\cdot)\|^2_{H^4(0,L)} + \lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}\left(\frac{1}{\lambda\eta}|v_t|^2 + \lambda^7 s^8\eta^7|v|^2\right) dxdt. \tag{59}$$

Part 3: *Carleman estimate for parabolic operators.* Since $z(\sigma) \in W^{1,\infty}(0,T;W^{1,\infty}(0,L))$ and $u \in H^1(0,T;H^4 \cap H^2_0(0,L))$, we have that $(fR_t)_{xx} - h \in L^2(0,T;L^2(0,L))$. Therefore, we can use the Carleman estimate given by Proposition 3, with $q_3 = 2\sigma_x$, $q_2 = \sigma_{xx} + \gamma$, $q_1 = z(\sigma)$ and $q_0 = z_x(\sigma)$, on the second term of the right-hand side of (59) to obtain

$$\begin{aligned}\frac{1}{C}N(f) &\leq C_1\|u(T_0,\cdot)\|^2_{H^4(0,L)} + C_2\left(s^{-4}\|u_{xx}(\cdot,0)\|^2_{H^1(0,T)} + \lambda^{-2}s^{-8}\|u_{xxx}(\cdot,0)\|^2_{H^1(0,T)}\right)\\ &\quad + \lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}|(fR_t)_{xx}|^2 dxdt + \lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}|h|^2 dxdt,\end{aligned} \tag{60}$$

with $C_2 := \|e^{-2\lambda\phi(\cdot,0)}\eta^3(\cdot,0)\|_{L^\infty(0,T)}$. Note that $z_t(\sigma) \in L^\infty(0,T;W^{1,\infty}(0,L))$ leads us to

$$\frac{1}{C}\iint_{Q_T} e^{-2\lambda\phi}|h|^2 dxdt \leq \lambda^{-5}s^{-11}\iint_{Q_T} e^{-2\lambda\phi}\left(\lambda^7 s^8\eta^7|u|^2 + \lambda^5 s^6\eta^5|u_x|^2\right) dxdt. \tag{61}$$

The fact that $(fR)_{xx} \in L^2(0,T;L^2(0,L))$ allows us to use, once again, the Carleman estimate given by Proposition 3, with $q_3 = 2\sigma_x$, $q_2 = \sigma_{xx} + \gamma$, $q_1 = z(\sigma)$ and $q_0 = z_x(\sigma)$, on the right-hand side of (61) to obtain

$$\begin{aligned}\frac{1}{C}\iint_{Q_T} e^{-2\lambda\phi}|h|^2 dxdt &\leq C_2\left(\lambda^{-2}s^{-8}\|u_{xx}(\cdot,0)\|^2_{L^2(0,T)} + \lambda^{-4}s^{-12}\|u_{xxx}(\cdot,0)\|^2_{L^2(0,T)}\right)\\ &\quad + \lambda^{-5}s^{-11}\iint_{Q_T} e^{-2\lambda\phi}|(fR)_{xx}|^2 dxdt.\end{aligned} \tag{62}$$

Plugging (62) in (60) gives us

$$\begin{aligned}\frac{1}{C}N(f) &\leq C_1\|u(T_0,\cdot)\|^2_{H^4(0,L)} + C_2\left(s^{-4}\|u_{xx}(\cdot,0)\|^2_{H^1(0,T)} + \lambda^{-2}s^{-8}\|u_{xxx}(\cdot,0)\|^2_{H^1(0,T)}\right)\\ &\quad + \lambda^{-3}s^{-7}\iint_{Q_T} e^{-2\lambda\phi}\left(|(fR_t)_{xx}|^2 + |(fR)_{xx}|^2\right) dxdt.\end{aligned} \tag{63}$$

Considering that $e^{-2\lambda\phi(t,x)} \leq e^{-2\lambda\phi(T_0,x)}$ holds for every $(t,x) \in Q_T$ and $R \in H^1(0,T;H^4(0,L))$, we can absorb the last term of the right-hand side of (63) with $N(f)$ by considering $s_0 > 0$ large, $s \geq s_0$ and $\lambda \geq e^{2s\|\beta\|_{L^\infty(0,L)}}$. Finally, the fact that $e^{-2\lambda^2/\alpha(T_0)} \leq e^{-2\lambda\phi(T_0,x)}$ for every $x \in (0,L)$ allows us to conclude that $\|f\|^2_{H^2(0,L)} \leq CN(f)$, from which the first inequality in (6) follows.

Part 4: *Conclusion.* The second inequality in (6) follows by using the continuous injections $H^1(0,T;H^4(0,L)) \hookrightarrow L^\infty(0,T;H^4(0,L))$ and $H^1(0,T;H^4(0,L)) \hookrightarrow H^1(0,T;W^{3,\infty}(0,L))$ on the first inequality in (6). The proof of Theorem 2 is complete. $\square$

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